Canonical Models of (Mixed) Shimura Varieties and Automorphic Vector Bundles

J.S. MILNE

Partially supported by NSF.

Automorphic Forms, Shimura Varieties, and L-Functions

ISBN 0-12-176651-9

Introduction.

This article surveys what is known to be true, or is conjectured, concerning the rationality properties over $\mathbb{Q}$ of automorphic functions, holomorphic automorphic forms, and the Fourier-Jacobi series of automorphic forms.

The first chapter reviews the theory of abelian varieties with potential complex multiplication over $\mathbb{Q}$ and the motives that are built out of them. The constructions and results in this chapter are the basis of the statements in the succeeding chapters.

The second chapter reviews the definition and basic properties of Shimura varieties, and then states the main results: every Shimura variety has a canonical model over its reflex field, and the conjugate of the canonical model by an element of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ is again the canonical model of a Shimura variety.

Holomorphic automorphic forms can be interpreted as the sections of certain vector bundles, called automorphic vector bundles, on a Shimura variety. These bundles are defined in the Chapter III, and the main theorems for them, which parallel those for Shimura varieties, are stated. In particular, every automorphic vector bundle has a canonical model over a specific number field, and we can define a holomorphic automorphic form to be rational over a field if it is a section of the canonical model of the vector bundle over that field.

As one approaches the boundary of a Hermitian symmetric domain, Hodge structures degenerate into mixed Hodge structures, and as one approaches the boundary of a Shimura variety, abelian varieties degenerate into one-motives. The theories of mixed Hodge structures and of one-motives are reviewed in Chapter IV.

In contrast to the Baily-Borel compactification of a Shimura variety, the method of toroidal compactification provides smooth compactifications of Shimura varieties. In Chapter V we describe these compactifications, and suggest how the various isomorphisms constructed in Chapters II and III should extend to the compactified varieties.

The study of the boundary of a Shimura variety suggests the introduction of a new object, generalizing that of a Shimura variety, which we here call a mixed Shimura variety. These varieties are defined in Chapter VI, and we indicate there how the results in Chapters II and III should extend to them. To give the reader some idea of how the notion of a mixed Shimura variety relates to that of a Shimura variety, we list some of the objects attached to a Shimura variety and the corresponding object attached to a mixed Shimura variety:

SHIMURA VARIETY	MIXED SHIMURA VARIETY
bounded symmetric domain	Siegel domain (of the third kind)
Hodge structure	mixed Hodge structure
reductive group	algebraic group with a 3-step filtration
abelian variety	one-motive
motive	mixed motive.

Roughly speaking, everything that is true for Shimura varieties should also be true for mixed Shimura varieties. For example, it will probably turn out to be most natural to study Hasse-Weil zeta functions in the context of mixed Shimura varieties rather than Shimura varieties. Lest the reader fear an unending hierachy, I mention that the study of the boundary of a mixed Shimura variety leads only to mixed Shimura varieties, not to some higher order object.

In the last chapter, we give a formal-algebraic definition of Fourier-Jacobi series, and suggest a theory for them also over $\mathbb{Q}$.

The contents of the second and third chapters will eventually be part of a book that I am currently writing on Shimura varieties. Once the theory outlined in the last four chapters is complete, a second book will be appropriate. Lest the reader think that that will then be the end of the subject, I point out that the theory for a general Shimura variety will then be in roughly the same happy state as the theory for elliptic modular curves was at the time of the publication of Shimura's book, Shimura (1971b), and that 1971 was the start of an explosion of interest in elliptic modular curves that continues to this day.

One of my goals in this article has been to write out the implications of Deligne's vision that Shimura varieties should be thought of as moduli varieties of motives and mixed Shimura varieties as the moduli varieties of mixed motives. I wish to thank Deligne for his patient explanation of his ideas to me over the years, and I mention specifically that the definition of a mixed Shimura variety in Chapter VI and the formal-algebraic definition of Fourier-Jacobi series in Chapter VII were suggested to me by him.

In this article, I have not attempted to describe in detail the origins of theorems, but have largely confined myself to listing the most recent work. Thus it is appropriate to mention that most of the questions discussed in this article first arose in the work of Shimura, and were often answered by him (or his students) in key cases. See in particular his talks to the International Congresses (Shimura 1968, 1971a, 1978a).

Finally I wish to thank Don Blasius and Michael Harris for many enjoyable and illuminating discussions on these questions; also I would like to thank them, Greg Anderson, and Pierre Deligne for their comments on parts of earlier drafts of this manuscript.

Conventions. All vector spaces and locally free sheaves are of finite rank. We use the same letter for a vector bundle and its associated locally free sheaf of sections.

A variety Y is a geometrically reduced scheme of finite-type over a field (it is not necessarily connected). For a variety Y over a field k and a homomorphism $\sigma : k \hookrightarrow k'$, we write σY for $Y \times_{\mathrm{Spec}(k),\sigma} \mathrm{Spec}(k')$ (the polynomials defining σY are obtained from those defining Y by applying σ to their coefficients). When it is not necessary to mention σ, we write $Y_{k'}$ for σY.

The following construction will be often used: let G be an algebraic group over $\mathbb{Q}$ acting on a variety Y on the left, and let P be a right principal homogeneous space for G; then $P \times^G Y$, the variety obtained from Y "by twisting by P", is the variety over $\mathbb{Q}$ such that, as a $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$-set,

$$(P \times^G Y)(\mathbb{Q}^{\mathrm{al}}) = P(\mathbb{Q}^{\mathrm{al}}) \times Y(\mathbb{Q}^{\mathrm{al}})/\sim, \ (pg, y) \sim (p, gy), \ g \in G(\mathbb{Q}^{\mathrm{al}}).$$

For an algebraic group G over $\mathbb{R}$, $G(\mathbb{R})^+$ is the identity component of the topological group $G(\mathbb{R})$ and $G(\mathbb{R})_+$ is the inverse image of $G^{\mathrm{ad}}(\mathbb{R})^+$ in $G(\mathbb{R})$; also $G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$ and $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$. An algebraic group is said to be simple when all its proper normal closed subgroups are finite. When an algebraic group G is defined over a field k, then all statements are relative to k; for example, "simple" means "k-simple", subgroups are defined over k, and representations take values in k-vector spaces.

When k is a field, k^{al} is an algebraic closure of k, k^{sep} is a separable algebraic closure, and k^{ab} is the maximal abelian extension of k. We always take $\mathbb{Q}^{\mathrm{al}}$ to be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

For a number field E, $\mathbb{A}_E$ is the ring of adèles of E and $\hat{E}$ the ring of finite adèles. We write $\mathbb{A}$ for $\mathbb{A}_{\mathbb{Q}}$, $\mathbb{A}_f$ for $\hat{\mathbb{Q}}$, and $\mathbb{A}'$ for $\mathbb{C} \times \mathbb{A}_f$. The reciprocity law $\mathrm{rec}_E : \mathbb{A}_E^{\times} \to \mathrm{Gal}(E^{\mathrm{ab}}/E)$ is normalized so that a local uniformizing element maps to the inverse of the usual (number-theorists) Frobenius automorphism. Complex conjugation is denoted by ι or by $a \mapsto \bar{a}$, and $[*\ ]$ is the equivalence class of $*$.

Except in Chapter V, the symbol T^F denotes the restriction of scalars (in the sense of Weil) of $\mathbb{G}_m$ from F to $\mathbb{Q}$.

When V is a vector space over a field k, and k' is an extension of k, we sometimes denote $V \otimes_k k'$ by $V(k')$ or $V_{k'}$.

I. Abelian varieties with complex multiplication

In this chapter we review the theory of abelian varieties with potential complex multiplication over $\mathbb{Q}$, the category of motives they generate, and their periods.

1. Tannakian categories.

The Pontryagin duality theorem allows one to recover a locally compact abelian group from its character group. Tannaka (1938) showed that a compact group can be recovered from the category of continuous finite-dimensional real representations of the group. The theory of Tannakian categories allows one to recover an affine group scheme from its category of finite-dimensional representations, and it gives an axiomatic characterization of the categories that arise in this fashion. It therefore provides a way of realizing certain abstractly defined categories as the category of representations of an affine group scheme.

A *tensor category* $(\mathbf{C}, \otimes)$ is a category $\mathbf{C}$ together with a functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ and sufficient constraints so that the tensor product of any finite unordered set of objects is well-defined up to a unique isomorphism. In particular, there is an *identity object* $\mathbf{1}$, defined to be the tensor product of the empty set of objects, which has the property that

$$X \otimes \mathbf{1} = X = \mathbf{1} \otimes X$$

for all objects X of $\mathbf{C}$.

A tensor category $(\mathbf{C}, \otimes)$ is said to be *abelian* when $\mathbf{C}$ is abelian and $\otimes$ is bi-additive. Then $k =_{\mathrm{df}} \mathrm{End}(\mathbf{1})$ is a commutative ring which acts on all objects of $\mathbf{C}$ in such a way that all morphisms of $\mathbf{C}$ are k-linear and $\otimes$ is bilinear; we call $(\mathbf{C}, \otimes)$ a *k-linear abelian tensor category* (in an alternative terminology, $(\mathbf{C}, \otimes)$ is called an *abelian tensor category with coefficents in k*). For example, $\mathbf{Vec}_k$ is a k-linear abelian tensor category.

A tensor category is said to be *rigid* if every object X of $\mathbf{C}$ has a dual $\check{X}$ and these duals have certain natural properties, for example, $\mathrm{Hom}(T \otimes \check{X}, Y) = \mathrm{Hom}(T, X \otimes Y)$.

A functor from one tensor category to a second is called a *tensor functor* if it carries tensor products into tensor products (including the identity object to the identity object). A *morphism of tensor functors $c : F \to F'$* is a morphism of functors commuting with tensor products, i.e., such that the diagrams

$$\begin{array}{ccc} \mathbf{1}' & \xrightarrow{\approx} & F(\mathbf{1}) \\ \| & & \downarrow{\scriptstyle c_1} \\ \mathbf{1}' & \xrightarrow{\approx} & F'(\mathbf{1}) \end{array}, \qquad \begin{array}{ccc} F(X \otimes Y) & \xrightarrow{\approx} & F(X) \otimes F(Y) \\ \downarrow{\scriptstyle c_{X \otimes Y}} & & \downarrow{\scriptstyle c_X \otimes c_Y} \\ F'(X \otimes Y) & \xrightarrow{\approx} & F'(X) \otimes F'(Y) \end{array}$$

commute. (The horizontal isomorphisms are part of the data that F and F' are tensor functors.)

Let k be a field. A *k-linear neutral Tannakian category* is a rigid k-linear abelian tensor category for which there exists an exact k-linear tensor functor $\omega : \mathbf{C} \to \mathbf{Vec}_k$. Such a functor is called a *fibre functor* for $(\mathbf{C}, \otimes)$. Since we shall never need to consider non-neutral Tannakian categories, from now "Tannakian category" means "neutral Tannakian category".

Example 1.1. For any affine group scheme G over a field k, the category $\mathbf{Rep}_k(G)$ of finite-dimensional representations of G on k-vector spaces is a k-linear Tannakian category with an obvious fibre functor, namely $(V, \xi) \mapsto V$. (An affine group scheme over k is an affine scheme G over k together with morphisms $G \times G \to G$ (multiplication), $G \to G$ (inverse), $\operatorname{Spec} k \to G$ (identity element) satisfying the usual axioms. Thus G is an algebraic group if it is of finite-type. Every affine group scheme is a projective limit of algebraic groups, and conversely every projective system of affine algebraic groups has an affine group scheme as limit.)

If ω is a fibre functor for the k-linear Tannakian category $(\mathbf{C}, \otimes)$ and R is a k-algebra, we define ω_R to be the tensor functor $X \mapsto \omega(X) \otimes_k R$ from $(\mathbf{C}, \otimes)$ to the category of R-modules. When ω' is a second fibre functor, $\mathcal{I}som^{\otimes}(\omega, \omega')$ denotes the functor from the category of k-algebras to that of sets,

$$R \mapsto \mathcal{I}som^{\otimes}(\omega_R, \omega'_R) \qquad \text{(isomorphisms of tensor functors).}$$

Also $\mathcal{A}ut^{\otimes}(\omega)$ denotes $\mathcal{I}som^{\otimes}(\omega, \omega)$.

THEOREM 1.2. *Let $(\mathbf{C}, \otimes)$ be a Tannakian category with fibre functor ω. The functor $\mathcal{A}ut^{\otimes}(\omega)$ is represented by an affine group scheme G over k, and ω defines an equivalence of tensor categories*

$$(\mathbf{C}, \otimes) \to (\mathbf{Rep}_k(G), \otimes).$$

If ω' is a second fibre functor, then the functor $\mathcal{I}som^{\otimes}(\omega, \omega')$ is represented by an affine scheme $P(\omega, \omega')$ which is a principal homogeneous space for G. The affine group scheme G' representing $\mathcal{A}ut^{\otimes}(\omega')$ is the inner form of G obtained from G by twisting by $P(\omega, \omega')$.

PROOF: See for example Deligne and Milne (1982), 2.11, 3.2.

The picture to keep in mind when thinking of Tannakian categories is the following. Let X be a connected topological manifold, and let $\mathbf{C}$ be the category of local systems of $\mathbb{Q}$-vector spaces on X (= locally constant sheaves of $\mathbb{Q}$-vector spaces). When endowed with its usual tensor structure, this category is Tannakian. The choice of a point x of X determines a fibre functor $\omega_x : \mathcal{V} \mapsto \mathcal{V}_x$ (stalk of $\mathcal{V}$ at x) for $\mathbf{C}$, and the fundamental group $\pi_1(X,x)$ acts on $\mathcal{V}_x$; moreover ω_x defines an equivalence from $(\mathbf{C},\otimes)$ to the tensor category of rational representations of the abstract group $\pi_1(X,x)$. If y is a second point, then the set $P(x,y)$ of paths from x to y (taken up to homotopy), is a principal homogeneous space for $\pi_1(X,x)$, and $\pi_1(Y,y)$ is the inner form of $\pi_1(X,x)$ obtained from $\pi_1(X,x)$ by twisting by $P(x,y)$.

Example 1.3. To give a grading on a vector space is the same as to give a representation of $\mathbb{G}_m$ on V: the grading $V = \oplus V^n$ corresponds to the representation for which $\mathbb{G}_m$ acts on V^n through the character $\chi_n = (t \mapsto t^n)$. The category of graded vector spaces over k has an obvious k-linear Tannakian structure, and our observation shows that the associated affine group scheme is $\mathbb{G}_m$.

Example 1.4. Let $\mathbf{C}$ be the category of continuous representations of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ on vector spaces over $\mathbb{Q}$. This is a $\mathbb{Q}$-linear Tannakian category with the forgetful functor as fibre functor. Write $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ as a limit $\varprojlim \mathrm{Gal}(K/k)$ of finite Galois groups, and give each group $\mathrm{Gal}(K/k)$ the structure of a constant algebraic group of dimension zero. Then $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ acquires the structure of a pro-algebraic group, and this is the affine group scheme attached to $\mathbf{C}$.

Remark 1.5. (a) A homomorphism $f : G \to G'$ of affine group schemes over k defines a tensor functor $F : \mathbf{Rep}_k(G') \to \mathbf{Rep}_k(G)$. Conversely, a tensor functor of k-linear Tannakian categories $F : (\mathbf{C},\otimes) \to (\mathbf{C}',\otimes)$ carrying a fibre functor ω' into a fibre functor ω defines a homomorphism of affine group schemes $f : Aut^{\otimes}(\omega') \to Aut^{\otimes}(\omega)$. Moreover, f is injective if and only if the image of F generates $\mathbf{Rep}_k(G)$ as a Tannakian category[1], and f is surjective if and only if F is fully faithful and the essential image is stable under the formation of subquotients.

[1] We say that a set of objects S in a Tannakian category $\mathbf{C}$ generates $\mathbf{C}$ if there is no full Tannakian subcategory of $\mathbf{C}$ containing all objects of S and their subquotients other than $\mathbf{C}$ itself.

(b) Let $(\mathbf{C}, \otimes)$ be a k-linear Tannakian category, and let k' be a finite separable extension of k. The category $\mathbf{C}_{k'}$ is defined to be the pseudo-abelian envelope[2] of the category whose objects are those of $\mathbf{C}$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathbf{C}_{k'}}(X, Y) = \mathrm{Hom}_{\mathbf{C}}(X, Y) \otimes_k k'.$$

It is a k'- linear Tannakian category. Any fibre functor ω of $\mathbf{C}$ extends in a natural way to a fibre functor ω' of $\mathbf{C}_{k'}$, and the affine group scheme attached to $(\mathbf{C}_{k'}, \omega')$ is $G_{k'}$.

Graded Tannakian categories.

Definition 1.6. A *grading* of a k-linear Tannakian category $\mathbf{C}$ can be described as either:

(a) a grading $X = \otimes_{m \in \mathbb{Z}} X^m$ on each object of $\mathbf{C}$ that depends functorially on X and is compatible with tensor products in the sense that $(X \otimes Y)^m = \oplus_{r+s=m} X^r \otimes Y^s$; or

(b) a central homomorphism $w : \mathbb{G}_m \to G$, $G = Aut^{\otimes}(\omega)$, for some fibre functor ω. Central means that the image is contained in the centre of G. Note that, by (1.2), the centre of G is independent of the choice of ω. A grading of $\mathbf{C}$ defines a grading on $\omega(X)$ for each object X and fibre functor ω; we have $\omega(X)^n = \omega(X^n)$, which is the subspace of $\omega(X)$ on which $w(z)$ acts as z^n.

Filtrations of $\mathbf{Rep}_k(G)$. Let V be a vector space. A homomorphism $\mu : \mathbb{G}_m \to GL(V)$ defines a filtration

$$\cdots \supset F^p V \supset F^{p+1} V \supset \cdots, \qquad F^p V = \oplus_{i \geq p} V^i,$$

of V, where $V = \oplus V^i$ is the grading defined by μ.

Let G be an algebraic group over a field k of characteristic zero. A homomorphism $\mu : \mathbb{G}_m \to G$ defines a filtration $F^{\cdot}$ on V for each representation (V, ξ) of G, namely, that corresponding to $\xi \circ \mu$. These filtrations are compatible with the formation of tensor products and duals, and they are exact in the sense that $V \mapsto Gr_F^{\cdot}(V)$ is exact.

[2] An additive category is *pseudo-abelian* or (*Karoubian*) if, for every morphism $p : X \to X$ such that $p^2 = p$, the kernel of $p - 1$ exists. For any additive category $\mathbf{C}$, there is a pseudo-abelian category $\mathbf{PC}$ and a functor $\mathbf{C} \to \mathbf{PC}$ that is universal among functors from $\mathbf{C}$ into pseudo-abelian categories. The objects of $\mathbf{PC}$ are pairs (X, p) with p as above, and the morphisms are defined so as to make (X, p) the image of p in the enlarged category.

Conversely, any functor $(V,\xi) \mapsto (V,F^{\cdot})$ from representations of G to filtered vector spaces compatible with tensor products and duals which is exact in this sense arises from a (nonunique) homomorphism $\mu : \mathbb{G}_m \to G$. We call such a functor a *filtration* $F^{\cdot}$ of $\mathbf{Rep}_k(G)$, and a homomorphism $\mu : \mathbb{G}_m \to G$ defining $F^{\cdot}$ is said to *split* $F^{\cdot}$. We write $Filt(\mu)$ for the filtration defined by μ.

For each p, we define F^pG to be the subgroup of G of elements acting as the identity map on $\oplus_i F^iV/F^{i+p}V$ for all representations V of G. Clearly F^pG is unipotent for $p \geq 1$, and F^0G is the semi-direct product of F^1G with the centralizer $Z(\mu)$ of any μ splitting $F^{\cdot}$.

PROPOSITION 1.7. *Let G be a reductive group over a field k of characteristic zero, and let $F^{\cdot}$ be a filtration of* $\mathbf{Rep}_k(G)$. *From the adjoint action of G on $\mathfrak{g} =_{\mathrm{df}} \mathrm{Lie}(G)$, we acquire a filtration of $\mathfrak{g}$.*

(a) F^0G is the subgroup of G respecting the filtration on each representation of G; it is a parabolic subgroup of G with Lie algebra $F^0\mathfrak{g}$.

(b) F^1G is the subgroup of F^0G acting trivially on the graded module $\oplus(F^pV/F^{p+1}V)$ associated with each representation of G; it is the unipotent radical of F^0G, and $\mathrm{Lie}(F^1G) = F^1\mathfrak{g}$.

(c) The centralizer $Z(\mu)$ of any μ splitting $F^{\cdot}$ is a Levi subgroup of F^0G; therefore, $Z(\mu) \xrightarrow{\approx} F^0G/F^1G$, and the composite $\bar{\mu}$ of μ with $F^0G \to F^0G/F^1G$ is central.

(d) Two cocharacters μ and μ' of G define the same filtration of G if and only if they define the same group F^0G and $\bar{\mu} = \bar{\mu}'$; μ and μ' are then conjugate under F^1G.

PROOF: See Saavedra (1972), especially IV.2.2.5.

Remark 1.8. It is sometimes more convenient to work with ascending filtrations. To turn a descending filtration $F^{\cdot}$ into an ascending filtration $W_{\cdot}$, set $W_i = F^{-i}$; if μ splits $F^{\cdot}$ then $z \mapsto \mu(z)^{-1}$ splits $W_{\cdot}$. With this terminology, we have $W_0G = W_{-1}G \rtimes Z(\mu)$.

Notes. The essentials of the theory of Tannakian categories are due to Grothendieck. A full account of the theory can be found in Saavedra (1972) and a more succinct account in Deligne and Milne (1982). The paper Deligne (1989) fills an important gap in the theory of non-neutral Tannakian categories.

2. Hodge structures.

A *real Hodge structure* is a real vector space V together with a decomposition

$$V \otimes \mathbb{C} = \oplus V^{p,q}$$

such that the complex conjugate of $V^{p,q}$ is $V^{q,p}$, all p, q. The category of such structures has a natural Tannakian structure, and the affine group scheme attached to the category and the forgetful fibre functor is $\mathbb{S} =_{\mathrm{df}} \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. According to Deligne's convention, $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ acts on $V^{p,q}$ as multiplication by $z^{-p}\bar{z}^{-q}$. A Hodge structure is said to be of *weight* n if $p+q=n$ for all (p,q) with $V^{p,q} \neq 0$. The *type* of a Hodge structure is the set of pairs (p,q) for which $V^{p,q} \neq 0$.

The *Hodge filtration* defined by a Hodge structure is

$$\cdots \supset F^p \supset F^{p+1} \supset \cdots, \qquad F^p = \oplus_{r \geq p} V^{r,s}.$$

If V has weight n, then

$$\bar{F}^q = (\oplus_{s \geq q} \overline{V}^{s,r}) = \oplus_{s \geq q} V^{r,s} = \otimes_{r \leq n-q} V^{r,s},$$

and so $V_{\mathbb{C}}$ is the direct sum of F^p and $\bar{F}^q$ whenever $p+q=n+1$. Conversely, if $F^{\cdot}$ is a finite descending filtration of $V_{\mathbb{C}}$ such that $V_{\mathbb{C}} = F^p \oplus \bar{F}^q$ whenever $p+q=n+1$, then $F^{\cdot}$ defines a Hodge structure of weight n on $V_{\mathbb{C}}$ by the rule $V^{p,q} = F^p \cap \bar{F}^q$.

From now on, we shall regard a real Hodge structure as being a pair (V,h) consisting of a real vector space V and a homomorphism $h : \mathbb{S} \to GL(V)$. We identify $\mathbb{S}_{\mathbb{C}}$ with $\mathbb{G}_m \times \mathbb{G}_m$ in such a way that $\mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C})$ becomes $z \mapsto (z, \iota z)$. The Hodge filtration on V is then the descending filtration defined by $\mu_h : \mathbb{G}_m \to GL(V_{\mathbb{C}})$, $\mu_h(z) = h_{\mathbb{C}}(z,1)$, and the weight grading is defined by $w_h : \mathbb{G}_m \to GL(V)$, $w_h(r) = h(r^{-1})$, $r \in \mathbb{R}^\times$.

For any $k \subset \mathbb{R}$, a *Hodge k-structure* is a vector space V over k together with a Hodge structure on $V \otimes_k \mathbb{R}$ such that the weight grading is defined over k. The category of such structures is a k-linear Tannakian category $\mathbf{Hdg}_k$. A Hodge $\mathbb{Q}$-structure will simply be called a *Hodge structure*. The *Mumford-Tate group* $MT(V,h)$ of a Hodge structure is the smallest $\mathbb{Q}$-rational algebraic subgroup of $GL(V) \times \mathbb{G}_m$ such that $MT(V,h)_{\mathbb{C}}$ contains the image of $(\mu_h, 1) : \mathbb{G}_m \to GL(V) \times \mathbb{G}_m$. It is a connected subgroup of $GL(V) \times \mathbb{G}_m$.

Example 2.1. (a) For any smooth projective variety X over $\mathbb{C}$, Hodge theory provides $H^n(X(\mathbb{C}), \mathbb{Q})$ with a Hodge structure of weight

n. Since $H_n(X(\mathbb{C}), \mathbb{Q})$ is dual to $H^n(X(\mathbb{C}), \mathbb{Q})$, it acquires a Hodge structure of weight $-n$.

(b) Giving a Hodge structure of type $\{(-1,0),(0,-1)\}$ on a real vector space V corresponds to giving a complex structure on V: given the complex structure, define $h(z)$ to be multiplication by z; given the Hodge structure, define the complex structure by the isomorphism $V \to V_{\mathbb{C}}/F^0$.

(c) For each integer n, $\mathbb{Q}(n)$ denotes the vector space $(2\pi \mathrm{i})^n\mathbb{Q}$ with the Hodge structure of type $\{(-n,-n)\}$.

A *polarization* of a Hodge k-structure (V, h) of weight n is a morphism of Hodge structures $\psi : V(\mathbb{R}) \otimes V(\mathbb{R}) \to \mathbb{R}(-n)$ such that the real-valued form $(x, y) \mapsto (2\pi \mathrm{i})^n \psi(x, h(i)y)$ is symmetric and positive-definite. The Mumford-Tate group of a polarizable Hodge structure is reductive.

Example 2.2. For an abelian variety A over $\mathbb{C}$, $H_1(A, \mathbb{Q})$ is a polarizable Hodge structure of type $\{(0,-1),(-1,0)\}$, and $A \mapsto H_1(A, \mathbb{Q})$ defines an equivalence between the category of abelian varieties over $\mathbb{C}$, considered up to isogeny, and the category of polarizable Hodge structures of type $\{(0,-1),(-1,0)\}$. The *Mumford-Tate* group MT^A of A is defined to be the Mumford-Tate group of $H_1(A, \mathbb{Q})$.

Hodge structures of CM-type. A Hodge structure is said to be of *CM-type* if it is polarizable and its Mumford-Tate group is commutative (and hence a torus).

Example 2.3. A field E of finite degree over $\mathbb{Q}$ is said to be a *CM-field* if there is a nontrivial involution ι of E that becomes complex conjugation under every embedding $E \hookrightarrow \mathbb{C}$. A finite product of CM-fields is called a *CM-algebra.* An abelian variety A is said to have *complex multiplication* (or be of *CM-type*) if there is a faithful homomorphism $E \to \mathrm{End}(A) \otimes \mathbb{Q}$ (mapping 1 to 1) with E a CM-algebra of degree $[E : \mathbb{Q}] = 2\dim(A)$, and it is said to have *potential complex multiplication* if it acquires complex multiplication over some extension of the ground field. With these definitions, an abelian variety over $\mathbb{C}$ is of CM-type if and only if the Hodge structure $H_1(A, \mathbb{Q})$ is of CM-type.

The category of Hodge structures of CM-type is Tannakian. Let $\mathfrak{S}$ be the affine group scheme attached to it and the forgetful fibre functor. The functor sending a Hodge structure (V, h) to the real

Hodge structure $(V \otimes \mathbb{R}, h)$ defines a homomorphism $h_{\text{can}} : \mathbb{S} \to \mathfrak{S}_{\mathbb{R}}$, and hence a cocharacter μ_{can} of $\mathfrak{S}_{\mathbb{C}}$.

PROPOSITION 2.4. *(a) The group scheme $\mathfrak{S}$ is a pro-torus. The map*

$$\xi \mapsto n_\chi, \qquad n_\chi(\tau) = \langle \chi, \tau\mu_{\text{can}} \rangle ,$$

identifies the character group of $\mathfrak{S}$ with the group of all functions $n : \mathrm{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q}) \to \mathbb{Z}$ which factor through $\mathrm{Gal}(F/\mathbb{Q})$ for some CM-field F and which have the property that

$$n(\iota\sigma) + n(\sigma) = \textit{constant}.$$

(b) The pair $(\mathfrak{S}, \mu_{\text{can}})$ has the following universal property: for any torus T over $\mathbb{Q}$ and $\mu \in X_*(T)$ satisfying

$$(*) \quad (\tau - 1)(\iota + 1)\mu = 0 = (\iota + 1)(\tau - 1)\mu, \qquad \textit{all } \tau \in \mathrm{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q}),$$

there is a unique homomorphism $\rho_\mu : \mathfrak{S} \to T$ (defined over $\mathbb{Q}$) such that $(\rho_\mu)_{\mathbb{C}} \circ \mu_{\text{can}} = \mu$.

The pro-torus $\mathfrak{S}$ is called the *Serre group*, and the condition (*) is called the *Serre condition.*

Remark 2.5. (a) For a field F of finite degree over $\mathbb{Q}$, define $\mathfrak{S}^F$ to be the quotient of $T^F =_{\text{df}} \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ whose character group $X^*(\mathfrak{S}^F)$ is the subgroup of $X^*(T^F)$ of elements satisfying the Serre condition. The norm map induces a homomorphism $\mathfrak{S}^{F'} \to \mathfrak{S}^F$ for any F' containing F, and it is easily seen that $\mathfrak{S} = \varprojlim \mathfrak{S}^F$ (limit over $F \subset \mathbb{Q}^{\text{al}}$). In fact, it suffices to take the limit over all CM-fields $F \subset \mathbb{Q}^{\text{al}}$.

(b) Let $F \subset \mathbb{Q}^{\text{al}}$ be a finite Galois extension of $\mathbb{Q}$. The action of $\mathrm{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$ on T^F defined by its action on F induces an action of $\mathrm{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$ on $\mathfrak{S}^F$. In the limit we obtain an action of $\mathrm{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$ on $\mathfrak{S}$ (rational over $\mathbb{Q}$). There are therefore two distinct actions of $\mathrm{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$ on $\mathfrak{S}(\mathbb{Q}^{\text{al}})$: the first arises from the action of $\mathrm{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$ on $\mathfrak{S}$, and the second from its action on $\mathbb{Q}^{\text{al}}$.

Example 2.6. Let E be a CM-algebra. A *CM-type* for E is a subset Φ of $\mathrm{Hom}(E, \mathbb{C})$ such that $\mathrm{Hom}(E, \mathbb{C}) = \Phi \cup \iota\Phi$ (disjoint union). Let A be an abelian variety over $\mathbb{C}$ with complex multiplication $i : E \to \mathrm{End}(A) \otimes \mathbb{Q}$ by E. For $\sigma \in \mathrm{Hom}(E, \mathbb{C})$, write $\mathbb{C}_\sigma$ for $\mathbb{C}$ with E acting through σ. Then $\mathrm{Tgt}_0(A) \approx \prod_{\varphi \in \Phi} \mathbb{C}_\varphi$ with Φ a CM-type for E, and

(A, i) is said to be of *CM-type* (E, Φ). By assumption, $V = H_1(A, \mathbb{Q})$ is a free E-module of rank one, and we can regard T^E as a subtorus of $GL(V)$. Define $\mu_\Phi : \mathbb{G}_m \to (T^E)_\mathbb{C}$ to be the cocharacter such that

$$\sigma \circ \mu_\Phi = \begin{cases} 1 & \text{if } \sigma \in \Phi \\ 0 & \text{otherwise,} \end{cases}$$

and let h_Φ be the associated homomorphism $h_\Phi : \mathbb{S} \to (T^E)_\mathbb{R}$. When regarded as a homomorphism $\mathbb{S} \to GL(V_\mathbb{R})$, h_Φ is the representation of $\mathbb{S}$ defined by the Hodge structure on $H_1(A, \mathbb{Q})$.

Since μ_Φ satisfies the Serre condition, it determines a homomorphism $\rho_\Phi : \mathfrak{S} \to T^E \subset GL(V)$; ρ_Φ is the representation of $\mathfrak{S}$ defined by the CM-Hodge structure $H_1(A, \mathbb{Q})$.

The field of definition of μ_Φ (contained in $\mathbb{C}$) is called the *reflex field* $E^*(\Phi)$ of (E, Φ). For any number field $F \supset E^*(\Phi)$, μ_Φ defines a homomorphism N_Φ

$$T^F \xrightarrow{\mathrm{Res}_{F/\mathbb{Q}}(\mu_\Phi)} \mathrm{Res}_{F/\mathbb{Q}}(T^E) \xrightarrow{\mathrm{Norm}_{F/\mathbb{Q}}} T^E$$

called the *reflex norm.*

For any isomorphism $\sigma : E \to E'$ of CM-fields and automorphism τ of $\mathbb{Q}^{\mathrm{al}}$, $\tau\Phi\sigma^{-1}$ denotes the CM-type $\{\tau\phi\sigma^{-1} | \phi \in \Phi\}$ of E'; for any CM-field $E' \supset E$, Φ extends to a CM-type $\Phi' = \{\phi \in \mathrm{Hom}(E', \mathbb{Q}^{\mathrm{al}}) | \ \phi|E \in \Phi\}$. We shall need the following formulas:

$$\rho_\Phi \circ \tau = \rho_{\tau^{-1}\Phi}, \qquad \sigma \circ \rho_\Phi = \rho_{\Phi\sigma^{-1}}, \qquad N_{E'/E} \circ \rho_{\Phi'} = \rho_\Phi.$$

Hodge tensors. Let V be a Hodge structure. A *Hodge element* in V is an element of type $(0,0)$ in V. For example, the Hodge elements in $\check{V} \otimes W$ are precisely the elements corresponding to homomorphisms $V \to W$ that are morphisms of Hodge structures. According to the Hodge conjecture, the Hodge elements of $H^{2p}(X, \mathbb{Q}(p))$ should be linear combinations of the classes of algebraic cycles. A *Hodge tensor* of V is an element of type $(0,0)$ in

$$\mathcal{T}V =_{\mathrm{df}} \oplus V^{\otimes r} \otimes \check{V}^{\otimes s} \otimes \mathbb{Q}(m) \qquad (\text{sum over } (r, s, m) \in \mathbb{N} \times \mathbb{N} \times \mathbb{Z}).$$

We let $GL(V)$ act on $\mathcal{T}V$ through its actions on V and $\check{V}$, and we let $\mathbb{G}_m$ act on $\mathcal{T}V$ through its action on $\mathbb{Q}(1)$.

PROPOSITION 2.7. *The Mumford-Tate group of a Hodge structure (V, h) is the subgroup of $GL(V) \times \mathbb{G}_m$ of elements fixing all the Hodge tensors of V.*

PROOF: See Deligne (1982a), pp40-45.

COROLLARY 2.8. *Let $\mathbf{C}$ be the Tannakian subcategory of $\mathbf{Hdg}_{\mathbb{Q}}$ generated by V and $\mathbb{Q}(1)$. The affine group scheme attached by (1.2) to $\mathbf{C}$ and the forgetful fibre functor is $MT(V, h)$.*

PROOF: Since V and $\mathbb{Q}(1)$ generate $\mathbf{C}$, the affine group scheme is a subgroup of $GL(V) \times \mathbb{G}_m$, and it consists of those elements of $GL(V) \times \mathbb{G}_m$ that commute with all morphisms of Hodge structures. But every morphism of Hodge structures in $\mathbf{C}$ can be interpreted as a Hodge tensor of V.

Notes. The Mumford-Tate group was introduced in Mumford (1966), and the Serre group in Serre (1968), pII-2. They are discussed in more detail in Deligne (1982a), §3, and Milne and Shih (1982a).

3. Hodge cycles.

A theorem of Deligne shows that Hodge cycles on an abelian variety have some of the properties of algebraic cycles; in particular, it will enable us to define Hodge cycles on an abelian variety over any field of characteristic zero.

We review the first homology groups attached to an abelian variety A over a field k of characteristic zero.

When $k = \mathbb{C}$, we have the usual "Betti" homology group $H_B(A) =_{\mathrm{df}} H_1(A(\mathbb{C}), \mathbb{Q})$. This is a vector space of dimension $2\dim A$ over $\mathbb{Q}$, and, as we noted in §2, it has a Hodge structure of type $\{(-1, 0), (0, -1)\}$. For any field k and embedding $\tau : k \hookrightarrow \mathbb{C}$, we set $H_\tau(A) = H_B(\tau A)$. When k is a subfield of $\mathbb{C}$, we sometimes write $H_B(A)$ for $H_B(A_{\mathbb{C}})$.

For any choice of an algebraic closure k^{al} of k, we define the ℓ-adic homology group $H_\ell(A)$ to be the dual of the étale cohomology group $H^1_{\mathrm{et}}(A_{k^{\mathrm{al}}}, \mathbb{Q}_\ell)$. This is a vector space of dimension $2\dim A$ over $\mathbb{Q}_\ell$. In more down-to-earth terms, we could set $H_\ell(A) = T_\ell(A) \otimes \mathbb{Q}$, where $T_\ell(A)$ is the Tate module $\varprojlim A(k^{\mathrm{al}})_{\ell^n}$ of A. An embedding of k^{al} into an algebraically closed field K defines an isomorphism $H_\ell(A_{k^{\mathrm{al}}}) \to H_\ell(A_K)$; in particular, $\mathrm{Gal}(k^{\mathrm{al}}/k)$ acts on $H_\ell(A)$. We set $\mathbb{Q}_\ell(1) = T_\ell(\mathbb{G}_m) \otimes \mathbb{Q}$, and $\mathbb{Q}_\ell(n) = \mathbb{Q}_\ell(1)^{\otimes n}$, $n \in \mathbb{Z}$.

We define $H_{\mathrm{dR}}(A)$ to be the dual of the de Rham cohomology group $H^1_{\mathrm{dR}}(A) =_{\mathrm{df}} \mathbb{H}^1(A, \Omega^{\cdot}_{A/k})$. It is a vector space of dimension $2\dim A$

over k, and if $K \supset k$, then $H_{\mathrm{dR}}(A_K) = H_{\mathrm{dR}}(A) \otimes_k K$. We sometimes write $H_\infty(A)$ for $H_{\mathrm{dR}}(A)$.

When $k = \mathbb{C}$, there are canonical *comparison isomorphisms*

$$H_B(A) \otimes \mathbb{Q}_\ell \to H_\ell(A), \qquad H_B(A) \otimes \mathbb{C} \to H_{\mathrm{dR}}(A).$$

The second of these can be obtained as follows: the map

$$\gamma \mapsto (\omega \mapsto \int_\gamma \omega),$$

identifies $H_B(A) \otimes \mathbb{C}$ with the dual of the space of differential forms of the first or second kind on A, which equals $\check{H}^1_{\mathrm{dR}}(A) = H_{\mathrm{dR}}(A)$. Thus the map is defined by the periods of A.

We extend these notations as follows:

$$\begin{aligned}
\mathcal{T}_B(A) &= \mathcal{T}(H_B(A)) \qquad \text{(case that } k = \mathbb{C}\text{)};\\
\mathcal{T}_\tau(A) &= \mathcal{T}(H_B(\tau A)) \qquad \text{(where } \tau \text{ is an embedding of } k \text{ into } \mathbb{C}\text{)};\\
\mathcal{T}_\ell(A) &= \oplus H_\ell(A)^{\otimes r} \otimes \check{H}_\ell(A)^{\otimes s} \otimes \mathbb{Q}_\ell(m);\\
\mathcal{T}_\infty(A) &= \mathcal{T}_{\mathrm{dR}}(A) = \otimes H_{\mathrm{dR}}(A)^{\otimes r} \otimes \check{H}_{\mathrm{dR}}(A)^{\otimes s}\\
\mathcal{T}_f(A) &= \Pi' \mathcal{T}_\ell(A) \qquad \text{(restricted product over finite primes } \ell\text{)}.
\end{aligned}$$

When $k = \mathbb{C}$, the comparison isomorphisms extend to canonical isomorphisms

$$\mathcal{T}_B(A) \otimes \mathbb{Q}_\ell \to \mathcal{T}_\ell(A), \qquad \mathcal{T}_B(A) \otimes \mathbb{C} \to \mathcal{T}_{\mathrm{dR}}(A).$$

Thus, for any abelian variety A over k and inclusion $\tau : k^{\mathrm{al}} \hookrightarrow \mathbb{C}$, there are canonical maps

$$\mathcal{T}_B(\tau A) \to \mathcal{T}_\ell(\tau A) \leftarrow \mathcal{T}_\ell(A)$$

for each ℓ (including $\ell = \infty$).

When A is an abelian variety over $\mathbb{C}$, a Hodge tensor s for the Hodge structure $H_B(A)$ is called a *Hodge cycle* on A; thus s is an element of type $(0,0)$ in $\mathcal{T}_B(A)$. The images of s under the comparison isomorphisms are called the *local components* s_ℓ of s for each ℓ (including ∞).

Let A be an abelian variety over an algebraically closed field k. A family $(s_\ell)_\ell$ with $s_\ell \in \mathcal{T}_\ell(A)$ ($\ell = \infty$ included) is called a *Hodge* *cycle on A relative to $\tau : k \hookrightarrow \mathbb{C}$* if there is a Hodge cycle s on τA whose local components are the images of the s_ℓ in $\mathcal{T}_\ell(\tau A)$ for all ℓ. Equivalently, we can say that (s_ℓ) is a Hodge cycle on A relative to τ if

(a) $s_\infty \in F^0\mathcal{T}_\infty$;

(b) the image of (s_ℓ) in $\mathcal{T}_f(\tau A) \times \mathcal{T}_\infty(\tau A)$ lies in the $\mathbb{Q}$-subspace $\mathcal{T}_B(\tau A)$.

THEOREM 3.1. *Let A be an abelian variety over an algebraically closed field k of characteristic zero. If s is a Hodge cycle on A relative to one embedding $\tau : k \hookrightarrow \mathbb{C}$, then it is a Hodge cycle relative to every such embedding.*

PROOF: This is the main theorem of Deligne (1982a).

Of course, the theorem says nothing if there are no embeddings of k into $\mathbb{C}$. When k is an algebraically closed field of finite transcendence degree over $\mathbb{Q}$, we write $C_H(A)$ for the subspace of $\mathcal{T}_f(A) \times \mathcal{T}_\infty(A)$ of elements that are Hodge cycles relative to some embedding of k into $\mathbb{C}$. It is a vector space over $\mathbb{Q}$, and an inclusion $k \hookrightarrow K$ of algebraically closed fields of finite transcendence degree over $\mathbb{Q}$ induces an isomorphism $C_H(A) \to C_H(A_K)$. This remark allows us to define $C_H(A)$ for an abelian variety over any algebraically closed field K of characteristic zero: choose an algebraically closed subfield k of K of finite transcendence degree over Q such that A has a model A_k over k and set $C_H(A) = C_H(A_k)$.

An embedding $k \hookrightarrow K$ of algebraically closed fields defines a map $C_H(A) \to C_H(A_K)$. In particular, when A has a model A_0 over subfield k_0 of k, $\mathrm{Gal}(k/k_0)$ acts on $C_H(A)$. In this case, we define $C_H(A_0)$ to be the subspace of $C_H(A)$ of elements fixed by $\mathrm{Gal}(k/k_0)$.

Much of the above discussion extends to arbitrary smooth projective varieties X. In particular, it is possible to define the notion of a Hodge cycle on X relative to an embedding $\tau : k \hookrightarrow \mathbb{C}$ (see Deligne 1982a, §2), and it is reasonable to expect that (3.1) will hold also for X.

CONJECTURE 3.2. *For any smooth projective variety X over an algebraically closed field k of characteristic zero, a cycle s that is a Hodge cycle relative to one embedding $\tau : k \hookrightarrow \mathbb{C}$ will be a Hodge cycle relative to every such embedding.*

This conjecture is implied by the Hodge conjecture. In the absence of a proof of (3.2), Deligne makes the following definition: when X

is defined over an algebraically closed field k of finite transcendence degree over $\mathbb{Q}$, an *absolute Hodge cycle* on X is a cycle that is Hodge relative to *every* embedding $k \hookrightarrow \mathbb{C}$. The definition is extended to other ground fields by the same procedure as for Hodge cycles on abelian varieties. This gives a notion of an absolute Hodge cycle on any smooth projective variety over a field of characteristic zero, which, when the variety is an abelian variety, coincides with that of a Hodge cycle.

Remark 3.3. Let A be an abelian variety over $\mathbb{C}$. Proposition 2.7 provides the following description of MT^A: for any $\mathbb{Q}$-algebra R, $MT^A(R)$ is equal to the group of automorphisms $H_B(A) \otimes R$ fixing all elements of $C_H(A)$.

Notes. This section summarizes part of Deligne (1982a).

4. Motives.

Let k be a field of characteristic zero, and let $\mathbf{V}/k$ be a category of smooth projective varieties over k. The aim of the theory of motives is to attach to $\mathbf{V}/k$ a $\mathbb{Q}$-linear Tannakian category $\mathbf{Mot}/k$ and a "universal cohomology functor" $h : \mathbf{V}/k \to \mathbf{Mot}/k$ (see Saavedra (1972), VI.4).

Example 4.1. Let $\mathbf{V}_0/k$ be the category of varieties of dimension zero over k. For a variety $X = \operatorname{Spec} R$ of dimension zero and $\tau : k \hookrightarrow \mathbb{C}$, we have the (zero$^{\text{th}}$) cohomology groups,

$$H_\tau(X) = \operatorname{Hom}(X(\mathbb{C}), \mathbb{Q}),\ H_\ell(X) = \operatorname{Hom}(X(k^{\text{al}}), \mathbb{Q}_\ell),\ H_{\text{dR}}(X) = R.$$

Fix an algebraic closure k^{al} of k, and let $\mathbf{Art}/k$ be the Tannakian category defined in (1.4). For a representation $M = (V, \xi)$ of $\operatorname{Gal}(k^{\text{al}}/k)$, define

$$H_\tau(M) = V, \quad H_\ell(M) = V \otimes \mathbb{Q}_\ell, \quad H_{\text{dR}}(M) = (V \otimes k^{\text{al}})^{\operatorname{Gal}(k^{\text{al}}/k)}$$

(diagonal action). Set $hX = \operatorname{Hom}(X(k^{\text{al}}), \mathbb{Q})$ for X in $\mathbf{V}_0$; then $\mathbf{Art}/k$ is generated (as a Tannakian category) by the objects hX, and $H_*(hX) = H_*(X)$ for $* = \tau$, ℓ, or dR. Thus $h : \mathbf{V}_0 \to \mathbf{Art}/k$ is the universal cohomology functor for $\mathbf{V}_0/k$. The objects of $\mathbf{Art}/k$ are called *Artin motives.*

Unfortunately, not enough is known about algebraic cycles to construct a Tannakian category of motives for all varieties using them[3].

[3] Surprisingly, the difficulty is in adjusting the commutativity constraint (the functorial isomorphism $X \otimes Y \approx Y \otimes X$). For this one needs to use Grothendieck's "standard conjectures"–see Saavedra (1972), VI.4.

Instead, we use Hodge cycles. Assume k is algebraically closed, and let $\mathbf{V}/k$ be the category of abelian varieties over k. If A and B are objects of $\mathbf{V}/k$, define $\mathrm{Hom}(hA, hB)$ to the set of families $(f_\ell : H_\ell(A) \to H_\ell(B))_\ell$ ($\ell = \infty$ included) such that, when we regard f_ℓ as an element of $\check{H}_\ell(A) \otimes H_\ell(B) \subset \mathcal{T}_\ell(A \times B)$, then $(f_\ell)_\ell$ is a Hodge cycle on $A \times B$. Define $\mathbf{CV}/k$ to be the category with objects hA, one for each $A \in \mathrm{Ob}(\mathbf{V}/k)$, and the morphisms just defined. Adjoin the images of projectors p to the set of objects of $\mathbf{CV}/k$, and so embed $\mathbf{CV}/k$ into its pseudo-abelian envelope $\mathbf{CV}^+/k$ (cf. 1.5a). Next adjoin to $\mathbf{CV}^+/k$ all powers of the Tate motive $\mathbb{Q}(1)$. Finally modify the commutativity constraint (the identification of $M \otimes N$ with $N \otimes M$) to obtain the category $\mathbf{AV}/k$ of motives of abelian varieties over k (for the details, see Deligne and Milne 1982, §6).

THEOREM 4.2. *The category $\mathbf{AV}/k$ is a semisimple $\mathbb{Q}$-linear Tannakian category. It is generated (as a Tannakian category) by the motives hA with A an abelian variety over K. The functors H_τ, H_ℓ, and H_{dR} on $\mathbf{V}/k$ extend to $\mathbf{Mot}/k$, as do the comparison isomorphisms.*

Variants 4.3. (a) Drop the condition that k is algebraically closed, and take $\mathbf{V}/k$ to be the category of abelian varieties and varieties of dimension zero over k. We then obtain a semisimple $\mathbb{Q}$-linear Tannakian category $\mathbf{AV}/k$ with the properties in (4.2) except that $\mathbf{AV}/k$ is now generated by the motives of abelian varieties and the Artin motives.

(b) Drop the condition that k is algebraically closed, and take $\mathbf{V}/k$ to be the category of all smooth projective varieties over k. Replace "Hodge cycle" with "absolute Hodge cycle" in the definition of $\mathbf{CV}/k$. We then obtain a semisimple $\mathbb{Q}$-linear Tannakian category $\mathbf{Mot}/k$, the *category of motives* over k, with the properties in (4.2), except that it is now generated by the motives of smooth projective varieties.

PROPOSITION 4.4. *The functor $H_B : \mathbf{AV}/\mathbb{C} \to \mathbf{Hdg}_{\mathbb{Q}}$ is fully faithful.*

PROOF: In this case, $\mathrm{Hom}(hA, hB)$ consists of the maps $H_B(A) \to H_B(B)$ given by Hodge tensors. These are morphisms of Hodge structures.

Motives of CM-type. Define $\mathbf{CM}/k$ to be the Tannakian subcategory of $\mathbf{AV}/k$ generated by the motives of abelian varieties of potential CM-type over k and the Artin motives. Objects of $\mathbf{CM}/k$ will be called *motives of CM-type* or *CM-motives* over k.

PROPOSITION 4.5. *The functor $H_B : \mathbf{CM}/\mathbb{C} \to \mathbf{Hdg}_{\mathbb{Q}}$ is fully faithful, with essential image the category of Hodge structures of CM-type. Therefore the affine group scheme attached to the Tannakian category $\mathbf{CM}/\mathbb{C}$ and the Betti fibre functor is the Serre group $\mathfrak{S}$.*

PROOF: That H_B is fully faithful follows from (4.4). If we let $\mathfrak{S}'$ be the affine group scheme attached to $\mathbf{CM}/\mathbb{C}$, then (1.5a) shows that there is a surjective homomorphism $\mathfrak{S} \to \mathfrak{S}'$. To prove that this homomorphism is injective, it suffices to show that the intersection of the kernels of the homomorphisms $\rho_A : \mathfrak{S} \to GL(H_B(A))$, A an abelian variety A of CM-type over $\mathbb{C}$, is trivial. This follows from the next lemma.

LEMMA 4.6. *Let $F \subset \mathbb{Q}^{\mathrm{al}}$ be a CM-field, Galois over $\mathbb{Q}$. The intersection of the kernels of the homomorphisms (see 2.6) $\rho_\Phi : \mathfrak{S}^F \to T^F$ defined by the CM-types Φ on F is trivial.*

PROOF: It suffices to show that $X^*(\mathfrak{S}^F)$ is generated by the images of the maps $X^*(\rho_\Phi) : X^*(T^F) \to X^*(\mathfrak{S}^F)$. But, by definition, $X^*(\mathfrak{S}^F)$ consists of the sums $\sum n(\sigma)\,\sigma$, $\sigma \in \mathrm{Hom}(F, \mathbb{C})$, with $n(\sigma) + n(\iota\sigma)$ constant, and one sees easily that the image of $X^*(\rho_\Phi)$ contains $\sum_{\varphi\in\Phi} \varphi$. Thus the proof is an easy combinatorial exercise (see Lang 1983, p175).

The functor $A \mapsto A_{\mathbb{C}}$ defines an equivalence between the category of abelian varieties of CM-type over $\mathbb{Q}^{\mathrm{al}}$ and the corresponding category over $\mathbb{C}$. Thus the base-change functor $\mathbf{CM}/\mathbb{Q}^{\mathrm{al}} \to \mathbf{CM}/\mathbb{C}$ is an equivalence of categories, and the affine group scheme attached to $\mathbf{CM}_{\mathbb{Q}^{\mathrm{al}}}$ and the Betti fibre functor is again the Serre group $\mathfrak{S}$.

Notes. The concept of a motive is due to Grothendieck. The definition adopted in this article is a variant of his. Most of the material in this section is from Deligne and Milne (1982), §6.

5. The main theorem of complex multiplication.

Let (A, i) be an abelian variety with complex multiplication over $\mathbb{Q}^{\mathrm{al}}$. The theorem of Shimura and Taniyama (Lang 1983, p84) describes how those automorphisms of $\mathbb{Q}^{\mathrm{al}}$ fixing the reflex field of (A, i) act on the torsion points of A. Work of Deligne and Langlands extends the result to the full Galois group of $\mathbb{Q}^{\mathrm{al}}$ over $\mathbb{Q}$. In this section, we give a statement and proof of this result in terms of abelian varieties, and in the next section, we re-interpret it in terms of motives.

Definition of the Taniyama element $f_\Phi(\tau)$. Let E be a CM-field. For each $\sigma \in \mathrm{Hom}(E, \mathbb{Q}^{\mathrm{al}})$, choose an element $v_\sigma \in \mathrm{Hom}(E^{\mathrm{ab}}, \mathbb{Q}^{\mathrm{al}})$ in such a way that $v_\sigma|E = \sigma$ and $v_{\iota\sigma} = \iota v_\sigma$. For any $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, $\tau \circ v_\sigma$ and $v_{\tau\sigma}$ have the same action on elements of E, and so differ by an element of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$. For a CM-type Φ for E, define

$$V_\Phi(\tau) = \prod_{\varphi \in \Phi} v_{\tau\varphi}^{-1} \cdot \tau \circ v_\varphi \in \mathrm{Gal}(E^{\mathrm{ab}}/E).$$

It is easily checked that $V_\Phi(\tau)$ is independent of the choice of the elements v_σ.

The *cyclotomic character* $\chi_{\mathrm{cyc}} : \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to \hat{\mathbb{Z}}^\times$ is defined by the condition that $\sigma\zeta = \zeta^{\chi_{\mathrm{cyc}}(\sigma)}$ for every root of unity ζ in $\mathbb{Q}^{\mathrm{al}}$. With our conventions, $\mathrm{rec}_{\mathbb{Q}}(\chi_{\mathrm{cyc}}(\sigma)) = \sigma|\mathbb{Q}^{\mathrm{ab}}$.

PROPOSITION 5.1. *There is a unique element $F_\Phi(\tau) \in \hat{E}^\times/E^\times$ such that*

(a) $\mathrm{rec}_E(f_\Phi(\tau)) = V_\Phi(\tau)$, *and*

(b) $f_\Phi(\tau) \cdot \iota f_\Phi(\tau) = \chi_{\mathrm{cyc}}(\tau)E^\times$.

PROOF: See Tate (1981) (also Lang 1983, p168).

We call $f_\Phi(\tau)$ the *Taniyama element* for (E, Φ) and τ. With the notations of (2.6), we we have the following result.

PROPOSITION 5.2. *(a)* $f_\Phi(\sigma\tau) = f_{\tau\Phi}(\sigma) \cdot f_\Phi(\tau)$, $\sigma, \tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$.

(b) $\sigma f_\Phi(\tau) = f_{\Phi\sigma^{-1}}(\tau)$, *$\sigma$ an isomorphism* $E \to E'$, $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$.

(c) $f_\Phi(\iota) = 1$.

(d) If Φ' is the extension of Φ to $E' \supset E$, then $f_\Phi(\tau) = f_{\Phi'}(\tau)$ (in $\hat{E}'^\times/E'^\times$).

(e) If τ fixes E^*, then $f_\Phi(\tau) = N_\Phi(s) \cdot E^\times$ for any $s \in \hat{E}^*$ such that $\mathrm{rec}_{E^*}(s) = \tau|E^{\mathrm{ab}}$.

PROOF: See Tate (1981) (also Lang 1983, VII).

First statement of the main theorem. Let (A, i) be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$ of CM-type (E, Φ), and let $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$. Define τi to be the map $E \to \mathrm{End}(\tau A) \otimes \mathbb{Q}$, $a \mapsto \tau(\mathrm{i}(a))$. Then $(\tau A, \tau i)$ is an abelian variety of CM-type $(E, \tau\Phi)$.

THEOREM 5.3. *(Main theorem, first form). Let (A, i) be of CM-type (E, Φ). For each $f \in \hat{E}^{\times}$ representing $f_{\Phi}(\tau)$, there is a unique E-linear isomorphism $\alpha : H_B(A) \to H_B(\tau A)$ such that $\tau x = \alpha(fx)$ for all $x \in H_f(A)$.*

PROOF: We explain in (5.10) below how to obtain a stronger result.

Remark 5.4. (a) It is obvious that α is uniquely determined by the choice of f representing $f_{\Phi}(\tau)$, and that if f is replaced by af $(a \in E^{\times})$, then α must be replaced by αa^{-1}.

(b) Let α be as in the theorem, and let ψ be a polarization of (A, i), that is, ψ is a polarization of $H_B(A)$ such that $\psi(ax, y) = \psi(x, \bar{a}y)$ for $a \in E$. Then, for x, $y \in H_f(A)$

$$(\tau\psi)(\tau x, \tau y) = \tau(\psi(x, y)) = \chi_{\mathrm{cyc}}(\tau) \cdot \psi(x, y)$$

because $\psi(x, y) \in \mathbb{A}_f(1)$. Thus if α is as in the theorem, then

$$\chi_{\mathrm{cyc}}(\tau) \cdot \psi(x, y) = (\tau\psi)(f\alpha(x), f\alpha(y)) = (\tau\psi)(f\bar{f}\alpha(x), \alpha(y))$$

and so

$$\psi(cx, y) = (\tau\psi)(\alpha x, \alpha y),$$

with $c = \chi_{\mathrm{cyc}}(\tau)/f\bar{f} \in E^{\times}$.

Now assume that A has complex multiplication by the full ring of integers $\mathcal{O}_E$ of E. The choice of a basis element e_0 for $H_B(A)$ determines an isomorphism $E \to H_B(A)$, and hence an isomorphism $\mathbb{C}^{\Phi} = E \otimes_{\mathbb{Q}} \mathbb{R} \to H_B(A) \otimes \mathbb{R} = \mathrm{Tgt}_0(A)$ (see 2.6). On composing this with the exponential map $\mathrm{Tgt}_0(A) \to A(\mathbb{C})$, we obtain an $\mathcal{O}_E$-linear isomorphism $\theta : \mathbb{C}^{\Phi}/\mathfrak{a} \to A(\mathbb{C})$ for some ideal $\mathfrak{a}$ in E. Moreover, the choice of e_0 allows us to write a polarization ψ of (A, i) in the form

$$\psi(xe_0, ye_0) = 2\pi \mathrm{i}\, \mathrm{Tr}_{E/\mathbb{Q}}(tx\bar{y})$$

for some $t \in E$. The triple (A, i, ψ) is then said to be of *type* $(E, \Phi; \mathfrak{a}, t)$ with respect to the parametrization θ. The type determines (A, i, ψ) up to isomorphism. If e_0 is replaced by $a^{-1}e_0$, then θ is replaced by θa^{-1}, and (A, i, ψ) is of type $(E, \Phi; a\mathfrak{a}, t/a\bar{a})$ with respect to θa^{-1}.

COROLLARY 5.5. *Let (A, i, ψ) be a polarized abelian variety over $\mathbb{C}$ of CM-type $(E, \Phi; \mathfrak{a}, t)$ with respect to a parametrization $\theta : \mathbb{C}^{\Phi} \to A(\mathbb{C})$, and let τ be an automorphism of $\mathbb{C}$. For each $f \in \hat{E}^{\times}$ representing $f_{\Phi}(\tau|\mathbb{Q}^{\mathrm{al}})$, there is a unique parametrization $\theta' : \mathbb{C}^{\tau\Phi} \to (\tau A)(\mathbb{C})$ of τA such that:*

(a) $\tau(A,i,\psi)$ *has type* $(E,\tau\phi;f\mathfrak{a},t\chi_{\mathrm{cyc}}(\tau)/f\bar{f})$ *with respect to* θ';
(b) *the diagram*

$$\begin{array}{ccc} E\mathfrak{a} & \xrightarrow{\theta} & A(\mathbb{C})_{\mathrm{tors}} \\ \downarrow f & & \downarrow \tau \\ E/f\mathfrak{a} & \xrightarrow{\theta'} & (\tau A)(\mathbb{C})_{\mathrm{tors}}, \end{array}$$

commutes.

PROOF: If θ is defined by $e_0 \in H_B(A)$, take θ' to be the parametrization of τA defined by $\alpha(e_0) \in H_B(\tau A)$, where α is the map in the theorem.

Remark 5.6. If τ fixes the reflex field and $s \in \hat{E}$ is such that $\mathrm{rec}_E(s) = \tau|E^{\mathrm{ab}}$, then $N_\Phi(s) \in f_\Phi(\tau)$ by (5.2c) and (5.5) becomes the theorem of Shimura and Taniyama referred to earlier.

Definition of the universal Taniyama element $f(\tau)$. Let T be a torus over $\mathbb{Q}$. For any Galois splitting field L of T, we set

$$\wp(T) = (T(\hat{L})/T(L))^{\mathrm{Gal}(L/\mathbb{Q})}.$$

This is easily seen to be independent of the choice of L. Moreover, if

$$H^1(\mathbb{Q},T) \to \prod_{\ell \text{ finite}} H^1(\mathbb{Q}_\ell,T)$$

is injective, then $\wp(T) = T(\hat{\mathbb{Q}})/T(\mathbb{Q})$. In particular, $\wp(T^E) = \hat{E}^\times/E^\times$. Define

$$\wp(\mathfrak{S}) = \varprojlim \wp(\mathfrak{S}^F).$$

PROPOSITION 5.7. *There is a unique element* $f(\tau) \in \wp(\mathfrak{S})$ *such that for each CM-field* E *and type* Φ, $\rho_\Phi(f(\tau)) = f_\Phi(\tau)$ *in* $\wp(T^E) = \hat{E}^\times/E^\times$. *The map* $\tau \mapsto f(\tau)$ *is a continuous reversed one-cocycle for* $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ *with values in* $\wp(\mathfrak{S})$, *that is,* $f^F(\sigma\tau) = \tau^{-1}f^F(\sigma)\cdot f^F(\tau)$.

PROOF: The uniqueness follows from (4.6). It is possible to prove the existence of $f(\tau)$ by verifying compatibilities between the $f_\Phi(\tau)$ for different Φ, but I prefer use Langlands's original construction of $f(\tau)$.

Let F be a finite Galois extension of $\mathbb{Q}$ contained in $\mathbb{Q}^{\mathrm{al}}$. The Weil group $W_{F/\mathbb{Q}}$ of F fits into an exact commutative diagram,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{A}_F^\times/F^\times & \longrightarrow & W_{F/\mathbb{Q}} & \longrightarrow & \mathrm{Gal}(F/\mathbb{Q}) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{rec}_F} & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathrm{Gal}(F^{\mathrm{ab}}/F) & \longrightarrow & \mathrm{Gal}(F^{\mathrm{ab}}/\mathbb{Q}) & \longrightarrow & \mathrm{Gal}(F/\mathbb{Q}) & \longrightarrow & 1 \end{array}$$

in which all the vertical arrows are surjective (see Tate 1979). If we assume further that F is a totally imaginary, then $(F \otimes \mathbb{R})^\times$ is contained in the kernel of rec_F, and so we can divide out by it and its image in $W_{F/\mathbb{Q}}$ to obtain an exact commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{F}^\times/F^\times & \longrightarrow & W^f_{F/\mathbb{Q}} & \longrightarrow & \mathrm{Gal}(F/\mathbb{Q}) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{rec}_F} & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathrm{Gal}(F^{\mathrm{ab}}/F) & \longrightarrow & \mathrm{Gal}(F^{\mathrm{ab}}/\mathbb{Q}) & \longrightarrow & \mathrm{Gal}(F/\mathbb{Q}) & \longrightarrow & 1. \end{array}$$

For each $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, choose an element $\tilde{\tau} \in W^f_{F/\mathbb{Q}}$ whose image in $\mathrm{Gal}(F^{\mathrm{ab}}/\mathbb{Q})$ is $\tau|F^{\mathrm{ab}}$. Choose elements $w_\sigma \in W^f_{F/\mathbb{Q}}$, one for each $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$, such that $w_\sigma \mapsto \sigma$ and $w_{\iota\sigma} = \tilde{\iota} w_\sigma$. Then $w_{\tau\sigma}$ and $\tilde{\tau} w_\sigma$ have the same image in $\mathrm{Gal}(F/\mathbb{Q})$, and so $w_{\tau\sigma}^{-1} \cdot \tilde{\tau} w_\sigma \in \hat{F}^\times/F^\times$.

LEMMA 5.8. *If F is a CM-field and Φ is a CM-type for F, then*

$$f_\Phi(\tau) = \prod_{\varphi \in \Phi} w_{\tau\varphi}^{-1} \cdot \tilde{\tau} w_\varphi.$$

PROOF: Write $f'_\phi(\tau)$ for the right hand side of the equation. It is obvious that the image of $f'_\Phi(\tau)$ in $\mathrm{Gal}(F^{\mathrm{ab}}/F)$ is $V_\Phi(\tau)$. Moreover, $f'_\Phi(\tau) \cdot \iota f'_\Phi(\tau) = f'_\Phi(\tau) \cdot f'_{\Phi\iota}(\tau) = \mathrm{Ver}(\tilde{\tau})$ where Ver is the transfer map $(W^f_{F/\mathbb{Q}})^{\mathrm{ab}} \to \hat{F}^\times/F^\times$ defined by the inclusion at top-left of the above diagram. But $\mathrm{Ver}(\tilde{\tau}) = \chi_{\mathrm{cyc}}(\tau) \cdot F^\times$, and so $f'_\Phi(\tau)$ has the properties characterizing $f_\Phi(\tau)$.

The canonical cocharacter μ^F of $\mathfrak{S}^F$ is defined over F, and therefore gives rise to a homomorphism $R^\times \to \mathfrak{S}^F(R)$ for any F-algebra R. Define

$$f^F(\tau) = \prod_{\sigma \in \mathrm{Gal}(F/\mathbb{Q})} (\sigma^{-1}\mu^F)(w_{\tau\sigma}^{-1}\tilde{\tau} w_\sigma) \in \mathfrak{S}^F(\hat{F})/\mathfrak{S}^F(F).$$

LEMMA 5.9. *Let E be a CM-field and Φ a CM-type for E. Assume that F is large enough to contain all conjugates of E in $\mathbb{C}$. Then $\rho_\Phi(f^F(\tau)) = f_\Phi(\tau)$ as elements of $T^E(\hat{F})/T^E(F) \supset T^E(\hat{\mathbb{Q}})/T^E(\mathbb{Q}) = \hat{E}^\times/E^\times$.*

PROOF: Let $\rho : E \hookrightarrow F \subset \mathbb{Q}^{\mathrm{al}}$ be an embedding of E. Then ρ defines a character $\boldsymbol{\rho}$ of T^E, and it suffices to show that $\boldsymbol{\rho}(\rho_\Phi(f^F(\tau))) = \boldsymbol{\rho}(f_\Phi(\tau))$ in $\hat{F}^\times/F^\times$. First note that, by (5.2),

$$\boldsymbol{\rho}(f_\Phi(\tau)) = \rho(f_\Phi(\tau)) = f_{\Phi\rho^{-1}}(\tau) = f_{\Phi'}(\tau),$$

where Φ' is the CM-type on F extending the CM-type $\Phi\rho^{-1}$ on $\rho E \subset F$. Next

$$\begin{aligned}
\boldsymbol{\rho}(\rho_\Phi(f^F(\tau)) &= \boldsymbol{\rho}(\prod_\sigma \rho_\Phi(\sigma^{-1}\mu^F)(w_{\tau\sigma}^{-1}\tilde{\tau}w_\sigma)) && \text{(definition of } f^F(\tau))\\
&= \boldsymbol{\rho}(\prod_\sigma \sigma^{-1}(\rho_\Phi \circ \mu^F)(w_{\tau\sigma}^{-1}\tilde{\tau}w_\sigma)) && (\rho_\Phi \text{ defined over } \mathbb{Q})\\
&= \boldsymbol{\rho}(\prod_\sigma \sigma^{-1}(\mu_\Phi)(w_{\tau\sigma}^{-1}\tilde{\tau}w_\sigma)) && \text{(definition of } \rho_\Phi)\\
&= \prod_\sigma (\boldsymbol{\rho} \circ \sigma^{-1}(\mu_\Phi))(w_{\tau\sigma}^{-1}\tilde{\tau}w_\sigma)\\
&= \prod_\sigma (w_{\tau\sigma}^{-1}\tilde{\tau}w_\sigma)^{\langle \boldsymbol{\rho}, \sigma^{-1}(\mu_\Phi)\rangle}
\end{aligned}$$

where $\langle\,,\,\rangle$ is the usual pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$. But we have $\langle \boldsymbol{\rho}, \sigma^{-1}\mu_\Phi\rangle = \langle \boldsymbol{\sigma} \circ \boldsymbol{\rho}, \mu_\Phi\rangle$, and from the definition of μ_ϕ in (2.6), we see that $\langle \boldsymbol{\sigma} \circ \boldsymbol{\rho}, \mu_\Phi\rangle = 1$ if $\sigma\rho \in \Phi$, and is 0 otherwise. Therefore the last product is $\prod_{\sigma\in\Phi'} w_{\tau\sigma}^{-1}\tilde{\tau}w_\sigma$, which (5.8) shows to equal $f_{\Phi'}(\tau)$.

We now complete the proof of (5.7). The elements $f^F(\tau)$ have the following properties:

(a) $f^F(\tau)$ is independent of all choices;
(b) f^F is a reversed one-cocycle;
(c) $\sigma f^F(\tau) = f^F(\tau)$, all $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$;
(d) if $F' \supset F$, then $N_{F'/F}(f^{F'}(\tau)) = f^F(\tau)$.

Statement (a) follows from (4.6). The remainder can be proved by applying ρ_Φ to both sides and using (5.2) and the formulas in (2.6). Statements (a), (c), and (d) show that $f(\tau) =_{\mathrm{df}} (f^F(\tau))$ is a well-defined element of $\wp(\mathfrak{S})$. As $\rho_\Phi(f(\tau)) = f_\Phi(\tau)$, this completes the proof of the first statement in (5.7). The second statement follows from (b).

We call $f(\tau)$ the *(universal) Taniyama element.*

Statement of the main theorem. Let A be an abelian variety of CM-type over $\mathbb{Q}$. On applying the homomorphism $\rho_A : \mathfrak{S} \to MT^A$ to $f(\tau)$, we obtain an element $f_A(\tau) \in \wp(MT^A)$.

THEOREM 5.10. *(Main theorem of complex multiplication) Let F be a splitting field of MT^A. For each $f \in MT^A(\hat{F})$ representing $f_A(\tau)$, there is a unique F- linear isomorphism $\alpha : H_B(A) \otimes F \to H_B(\tau A) \otimes F$ such that*

(a) $\alpha(t) = \tau t$ for all Hodge cycles on A;
(b) $\tau x = \alpha(fx)$ for all $x \in H_f(A) \otimes F$.

Remark 5.11. (a) It is possible to replace A in the theorem with any CM-motive over $\mathbb{Q}$ – it makes sense to speak of Hodge cycles on a CM-motive, and we can define the Mumford-Tate group of a CM-motive to be the image of $\mathfrak{S}$ in $GL(H_B(M)) \times \mathbb{G}_m$. The proof we describe below also applies to this more general case.

(b) Endomorphisms of A are Hodge cycles on A, and so (a) implies that α commutes with the action of all endomorphisms of A.

(c) It is again obvious that α is uniquely determined by the choice of f representing $f_A(\tau)$, and that if f is replaced by af $(a \in MT^A(F))$, then α must be replaced with αa^{-1}.

(d) To see that (5.10) implies (5.3), let (A, i) be as in (5.3), and let f' represent $f_\Phi(\tau)$. Note that $MT^A \subset T^E$. The definition of $f(\tau)$ shows that there is an element $a \in T^E(F)$ such that $f' = af$. Then $\alpha' = \alpha \circ a^{-1}$ satisfies the conditions of (5.3).

Proof of the main theorem of complex multiplication. We first define an element $g(\tau)$ such that Theorem 5.10 holds (tautologically) with f replaced by g.

LEMMA 5.12. *Let A be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$ of CM-type, and let F be a splitting field for MT^A. There exists an F-linear isomorphism $\alpha : H_B(A) \otimes F \to H_B(\tau A) \otimes F$ such that $\alpha(t) = \tau t$ for all Hodge cycles t on A.*

PROOF: For any $\mathbb{Q}$-algebra R, let

$$P(R) = \{\alpha : H_B(A) \otimes R \to H_B(\tau A) \otimes R \,|\, \alpha(t) = \tau t, \text{ all } t \in C_H(A)\}.$$

From (3.3) it is obvious that P is a torsor for MT^A unless it is empty. The comparison isomorphisms show that $P(\mathbb{C}) \neq 0$. Because MT^A is a torus split by F, the cohomology class of P in $H^1(\mathbb{Q}, MT^A)$ becomes trivial in $H^1(F, MT^A)$, which means that $P(F)$ is nonempty.

Let (A, i) be of CM-type (E, Φ), and choose an element $\alpha \in P(\hat{F})$. We can regard α as an isomorphism $\alpha : H_f(A) \otimes F \to H_f(\tau A) \otimes F$ sending t to τt, for all Hodge cycles t. The map $x \mapsto \alpha^{-1}(\tau x)$ is an automorphism of $H_f(A) \otimes F$ fixing all Hodge cycles, and so (3.3) shows that it is multiplication by an element $g \in MT^A(\hat{F})$. Write $g_A(\tau)$ for the image of g in $MT^A(\hat{F})/MT^A(F)$. Then $g_A(\tau)$ is independent of the choice of α, and it is fixed under the action of $\mathrm{Gal}(F/\mathbb{Q})$. It therefore lies in $\wp(MT^A)$. For varying A, the elements $g_A(\tau)$ form a projective system. As $\mathfrak{S} = \varprojlim MT^A$, they define an element $g(\tau) \in \wp(MT^A)$. Obviously (5.10) becomes true when $f(\tau)$ is replace by $g(\tau)$, and so, to prove (5.10), it suffices to show that $f(\tau) = g(\tau)$. Let $e(\tau) = g(\tau)/f(\tau)$ and, for each CM-type (E, Φ), let $e_\Phi(\tau) = \rho_\Phi(e(\tau))$. The next two lemmas prove that $e_\Phi(\tau) = 1$.

LEMMA 5.13. *The elements $e_\Phi(\tau)$ have the following properties:*

(a) $e_\Phi(\sigma\tau) = e_{\tau\Phi}(\sigma) \cdot e_\Phi(\tau)$, $\tau_1, \tau_2 \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$.

(b) $\sigma e_\Phi(\tau) = e_{\Phi\sigma^{-1}}(\tau)$, σ an isomorphism $E \to E'$, $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$.

(c) $e_\Phi(\iota) = 1$.

(d) If $E' \supset E$ and Φ' is the extension of Φ to E', then $e_\Phi(\tau) = e_{\Phi'}(\tau)$.

(e) If $\tau\Phi = \Phi$, then $e_\Phi(\tau) = 1$.

(f) If $\sum n_i \Phi_i = 0$, then $\prod e_{\Phi_i}(\tau)^{n_i} = 1$.

PROOF: Parts (b), (d), and (f) are automatic consequences of the fact that $e_\Phi(\tau) = \rho_\Phi(e(\tau))$ for an $e(\tau)$ in $\wp(\mathfrak{S})$. Part (a) follows from the fact that $f(\tau)$ and $g(\tau)$, and hence $e(\tau)$, are reversed one-cocycles. Part (c) holds for both f_Φ and g_Φ. For (e) note that $\tau\Phi = \Phi$ if and only if τ fixes the reflex field, and so the theorem of Shimura and Taniyama (see 5.6) shows that in this case $g_\Phi(\tau) = N_\Phi(s) \cdot E^\times$ where s is such that $\mathrm{rec}_E(s) = \tau | E^{\mathrm{ab}}$. Therefore (5.2e) implies (e).

LEMMA 5.14. *Let $(e_\Phi(\tau))$ be a family of elements satisfying the conditions of (5.13). Then $e_\Phi(\tau) = 1$ for all Φ and τ.*

PROOF: See Deligne 1981 (also Lang 1983, VII.4).

Remark 5.15. If $f(\tau)$ is a reversed one-cocycle, then $\tau \mapsto \tau f(\tau)$ and $\tau \mapsto f(\tau^{-1})^{-1}$ are both one-cocycles. It would have been possible to work throughout with one-cocycles rather than reversed one-cocycles, but the reversed one-cocycles are more consistent with the notations used in the literature.

Notes. See the end of the next section.

6. CM-motives over $\mathbb{Q}$; the Taniyama group.

In this section we study $\mathbf{CM}/\mathbb{Q}$, the category of CM-motives over $\mathbb{Q}$. It is a semisimple $\mathbb{Q}$-linear Tannakian category with additional structure, to which the Tannakian formalism attaches certain objects.

(6.1a) To $\mathbf{CM}/\mathbb{Q}$ and the Betti fibre functor H_B, Theorem 1.2 attaches an affine group scheme $\mathfrak{T}$.

(6.1b) To the fully faithful tensor functor $\mathbf{Art}/\mathbb{Q} \hookrightarrow \mathbf{CM}/\mathbb{Q}$, (1.5a) attaches a surjective homomorphism $\pi : \mathfrak{T} \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$.

(6.1c) H_B is an essentially surjective functor from $\mathbf{CM}/\mathbb{Q}$ to the category of Hodge structures of CM-type; it therefore defines a injective homomorphism $i : \mathfrak{S} \to \mathfrak{T}$.

(6.1d) The action of $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ on $H_\ell(M)$ sends s_ℓ to τs_ℓ for each Hodge cycle s. Therefore, each $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ defines an automorphism $sp_\ell(\tau)$ of the fibre functor $H_B \otimes \mathbb{Q}_\ell$ whose image in $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ is τ. The map sp_ℓ is a homomorphism $sp_\ell : \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to \mathfrak{T}(\mathbb{Q}_\ell)$ which is continuous for the Krull and ℓ-adic topologies, and the product of the sp_ℓ's defines a homomorphism

$$sp : \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to \mathfrak{T}(\mathbb{A}_f).$$

PROPOSITION 6.2. *The sequence of affine group schemes*

$$1 \to \mathfrak{S} \xrightarrow{i} \mathfrak{T} \xrightarrow{\pi} \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to 1$$

is exact. In particular, i identifies $\mathfrak{S}$ with the identity component of $\mathfrak{T}$. Moreover, the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ *on $\mathfrak{S}$ defined by the sequence is that described in (2.5b).*

PROOF: See Deligne (1982b).

Symbolically, we have a diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathfrak{S} \arrow[r, "i"] & \mathfrak{T} \arrow[r, "\pi"] \arrow[d] & \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \arrow[r] \arrow[dl, "sp"] & 1. \\ & & \mathfrak{T}(\mathbb{A}_f) & & \end{tikzcd}$$

The group $\mathfrak{T}$, together with the structure (π, i, sp), is called the *Taniyama group.* A CM-motive M over $\mathbb{Q}$ corresponds to a representation $\rho : \mathfrak{T} \to GL(V)$; then $H_B(M) = V$, and its Hodge structure of

CM-type is determined by $\rho \circ i$; the ℓ-adic cohomology group $H_\ell(M)$ is $V \otimes \mathbb{Q}_\ell$ with $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ acting through $\rho \circ sp_\ell$; and M is an Artin motive if and only if ρ factors through π. The Taniyama group does not enable us to construct $H_{\mathrm{dR}}(M)$ from (V, ρ) (we discuss what is needed for this in the next section).

Remark 6.3. (a) It is possible to interpret the exact sequence in (6.2) in the following way: a representation ρ of $\mathfrak{S}$ determines a CM-motive M over $\mathbb{Q}^{\mathrm{al}}$; extending ρ to $\mathfrak{T}$ corresponds to giving a descent datum on M, and descent is effective for CM-motives.

(b) For each $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, $M \mapsto H_\tau(M) = H_B(\tau M)$ is a fibre functor for $\mathbf{CM}/\mathbb{Q}^{\mathrm{al}}$ with values in $\mathbf{Vec}_{\mathbb{Q}}$. Therefore $Isom(H_B, H_\tau)$ is a torsor for $\mathfrak{S}$. It is represented by ${}^\tau\mathfrak{S} =_{\mathrm{df}} \pi^{-1}(\tau)$.

An explicit description of $(\mathfrak{T}, \pi, i, sp)$. In this subsection, we let $(\mathfrak{T}, \pi, i, sp)$ denote any quadruple for which (6.2) is true. Let $\mathfrak{S}'$ be a quotient of $\mathfrak{S}$ of finite-type over $\mathbb{Q}$, and let $\mathfrak{T}'$ be the quotient of $\mathfrak{T}$ by the kernel of $\mathfrak{S} \to \mathfrak{S}'$:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathfrak{S} & \xrightarrow{i} & \mathfrak{T} & \xrightarrow{\pi} & \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathfrak{S}' & \xrightarrow{i'} & \mathfrak{T}' & \xrightarrow{\pi'} & \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) & \longrightarrow & 1.
\end{array}$$

If L is a finite Galois extension of $\mathbb{Q}$ (contained in $\mathbb{Q}^{\mathrm{al}}$) splitting $\mathfrak{S}'$, then $H^1(L, \mathfrak{S}') = 0$, and so each of the $\mathfrak{S}'$-torsors $\pi'^{-1}(\tau)$ has a point in L. Therefore, we can choose a section $a : \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to \mathfrak{T}'(L)$ to π'. Identify $\mathfrak{T}'(L)$ and $\mathfrak{T}'(\hat{\mathbb{Q}})$ with subgroups of $\mathfrak{T}'(\hat{L})$, and write

$$sp(\tau) = a(\tau) \cdot h'(\tau), \qquad h'(\tau) \text{ in } \mathfrak{S}'(\hat{L}).$$

The class of $h'(\tau)$ in $\mathfrak{S}'(\hat{L})/\mathfrak{S}'(L)$ is independent of the choice of $a(\tau)$.

LEMMA 6.4. *The map $h' : \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to \mathfrak{S}'(\hat{L})/\mathfrak{S}'(L)$ has the following properties:*

(a) h' is a reversed one-cocycle;

(b) $\sigma h'(\tau) = h'(\tau)$ for all $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$; thus $h'(\tau) \in \wp(\mathfrak{S}')$.

PROOF: Straightforward.

Recall that $\wp(\mathfrak{S}) = \varprojlim \wp(\mathfrak{S}')$. The h''s therefore define a continuous reversed one-cocycle $h : \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to \wp(\mathfrak{S})$.

PROPOSITION 6.5. *Every quadruple $(\mathfrak{T}, \pi, i, sp)$ satisfying the conditions of (6.2) defines a continuous reversed one-cocycle*

$$h : \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to \wp(\mathfrak{S})$$

and h determines the quadruple $(\mathfrak{T}, \pi, i, sp)$ uniquely up to a unique isomorphism; moreover every reversed one-cocycle arises from a quadruple $(\mathfrak{T}, \pi, i, sp)$ satisfying the conditions of (6.2).

PROOF: We have already shown how to derive h from the quadruple. Obviously h determines the isomorphism class of $(\mathfrak{T}, \pi, i, sp)$, but such a quadruple is rigid: any automorphism of $\mathfrak{T}$ compatible with (π, i, sp) is the identity map. Finally, it is straightforward to construct the quadruple out of h (see for example Milne and Shih 1982a, §2).

The next result provides an explicit description of the Taniyama group.

THEOREM 6.6. *The reversed one-cocycle corresponding to the Taniyama group is $\tau \mapsto f(\tau)$, where $f(\tau)$ is the universal Taniyama element defined in §5.*

PROOF: Let h be the reversed one-cocycle corresponding to the Taniyama group. After the main theorem of complex multiplication (5.10) (more specifically, 5.14), we know that $f = g$, and so we have to prove that $h = g$. Let A be an abelian variety of CM-type over $\mathbb{Q}$, and let $h_A(\tau) = \rho_A(h(\tau))$. One sees immediately from their constructions that $h_A(\tau) = g_A(\tau)$ in $\wp(MT^A)$. Since $\mathfrak{S} = \varprojlim MT^A$, this proves the theorem.

Application to the zeta functions of CM-motives. It is possible to attach an L-series $L(\rho, s)$ to a complex representation $\rho : W_{\mathbb{Q}} \to GL(V)$ of the Weil group. Moreover, it is known that $L(\rho, s)$ extends to a meromorphic function on the whole complex plane and satisfies a functional equation (see Tate 1979). These L-series generalize both Hecke L-series and Artin L-series, and so are usually referred to as *Artin-Hecke* L-series.

PROPOSITION 6.7. *There is a homomorphism $W_{\mathbb{Q}} \to \mathfrak{T}(\mathbb{C})$ making the following diagram commute:*

$$\begin{tikzcd}
 & & & W_{\mathbb{Q}} \arrow[dl] \arrow[d] & \\
1 \arrow[r] & \mathfrak{S}(\mathbb{C}) \arrow[r] & \mathfrak{T}(\mathbb{C}) \arrow[r] & \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \arrow[r] & 1.
\end{tikzcd}$$

PROOF: See for example Milne and Shih (1982a), 3.17.

THEOREM 6.8. *For any CM-motive M, the system of ℓ-adic representations $H_\ell(M)$ is strictly compatible (in the sense of Serre 1968). Therefore the zeta function of M is defined, and it is an Artin-Hecke L-series.*

PROOF: This follows directly from (6.7) (see Schappacher 1988).

Remark 6.9. There is in fact a one-to-one correspondence between the set of isomorphism classes of CM-motives with coefficients in $\mathbb{Q}^{\mathrm{al}}$ defined over $\mathbb{Q}$ and the set of isomorphism classes of representations of $W_{\mathbb{Q}}$ of type A_0.

Algebraic Hecke characters. Let F be a finite extension of $\mathbb{Q}$, and let ${}^F\mathfrak{T}$ be the inverse image of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/F)$ in $\mathfrak{T}$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathfrak{S} & \longrightarrow & {}^F\mathfrak{T} & \longrightarrow & \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/F) & \longrightarrow & 1 \\ & & \| & & \downarrow\cap & & \downarrow\cap & & \\ 1 & \longrightarrow & \mathfrak{S} & \longrightarrow & \mathfrak{T} & \longrightarrow & \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) & \longrightarrow & 1. \end{array}$$

Then ${}^F\mathfrak{T}$ is the affine group scheme attached to the $\mathbf{CM}/F$. A homomorphism $\chi : {}^F\mathfrak{T} \to T^E$ is called an *algebraic Hecke character* for F with values in E. The restriction of χ to $\mathfrak{S}$ is the *infinity type* of χ, and for each prime ℓ,

$$sp_\ell \circ \chi : \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to T^E(\mathbb{Q}_\ell) = (E \otimes \mathbb{Q}_\ell)^\times$$

is the ℓ-adic representation attached to χ.

Notes. The reversed one-cocycle f (the universal Taniyama element of §5) was defined by Langlands in order to be able to describe the conjugate of a Shimura variety (Langlands 1979). Deligne recognized that it should define the affine group scheme attached to $\mathbf{CM}/\mathbb{Q}$, and proved that this was the case in (Deligne 1982b). The implications of Langlands's construction for abelian varieties of CM-type were also made explicit in Milne and Shih (1981a). Tate gave the construction of $f_\Phi(\tau)$ described in the first subsection of §5 in (Tate 1981). The relation between the constructions of Langlands and Tate has not previously been elucidated in print.

Deligne first proved the main theorem of complex multiplication in the form (6.6), expressing it in terms of extensions (Deligne 1982b). He then re-expressed the proof in terms of the functions e_Φ, as we did in §5 (Deligne 1981). It is also possible to express the proof directly in terms of the function e (Milne 1981).

7. Periods of CM-motives.

After the last section, it remains to describe the de Rham fibre functor on **CM**/$\mathbb{Q}$. This is again a $\mathbb{Q}$-linear fibre functor, and so (see 1.2) $\mathfrak{P} = \mathit{Isom}^{\otimes}(H_B, H_{\mathrm{dR}})$ is a principal homogeneous space for $\mathfrak{T}$ – we call it the *period torsor*. The comparison isomorphisms $H_B(M)\otimes\mathbb{C} \to H_{\mathrm{dR}}(M \otimes \mathbb{C})$ preserve Hodge cycles, and so define a canonical point $p \in \mathfrak{P}(\mathbb{C})$.

When M is the CM-motive corresponding to the representation $\rho : \mathfrak{T} \to GL(V)$ of $\mathfrak{T}$, $\mathfrak{P}$ enables us to construct the de Rham cohomology of M : $H_{\mathrm{dR}}(M) = \mathfrak{P} \times^{\mathfrak{T},\rho} V$. The point p gives us the comparison isomorphism $H_B(M) \otimes \mathbb{C} \to H_{\mathrm{dR}}(M_{\mathbb{C}})$.

The next conjecture, which is a variant of a conjecture of Grothendieck, predicts that the only restrictions on the transcendence of the periods of CM-motives come from Hodge cycles.

CONJECTURE 7.1. *The point p is generic in the sense that it is not contained in the set of complex points of any proper $\mathbb{Q}$-rational subscheme of $\mathfrak{P}$.*

Remark 7.2. Let $\mathbb{Q}[\mathfrak{P}]$ be the affine algebra of $\mathfrak{P}$. Then the point p corresponds to a homomorphism $\mathbb{Q}[\mathfrak{P}] \to \mathbb{C}$, and the conjecture is equivalent to this map being injective (because $\mathfrak{P}$ is irreducible).

Remark 7.3. Let $F \subset \mathbb{Q}^{\mathrm{al}}$ be a number field. On **CM**/F, H_{dR} is an F-linear fibre functor, and so the comparison isomorphism gives us a period torsor ${}^F\mathfrak{P}$ for $({}^F\mathfrak{T})_F =_{\mathrm{df}} {}^F\mathfrak{T} \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\, F$. One sees easily that ${}^F\mathfrak{P}$ is the inverse image of i under $\mathfrak{P}_F \to \mathrm{Hom}(F, \mathbb{Q}^{\mathrm{al}})$, where i is the given inclusion $F \hookrightarrow \mathbb{Q}^{\mathrm{al}}$. The canonical point p of $\mathfrak{P}(\mathbb{C})$ lies in ${}^F\mathfrak{P}(\mathbb{C})$.

Let $\chi : {}^F\mathfrak{T} \to T^E$ be an algebraic Hecke character for F with values in E. Then $\chi_*({}^F\mathfrak{P}) = \mathfrak{P}_\chi$ is a principal homogeneous space for $(T^E)_F$ with a distinguished complex point p_χ. As $H^1(F, T^E) = 0$, $\mathfrak{P}_\chi$ will have an F-rational point p_0, and any two such points differ by multiplication by an element of $T^E(F)$. Write $p_\chi = p_0 \cdot p(\chi)$; then $p(\chi)$ is a well-defined element of $(E\otimes F)^\times \backslash (E\otimes\mathbb{C})^\times$ called the *period* of χ. For example, if χ is the algebraic Hecke character attached to an abelian variety A over F with complex multiplication by E, then $p(\chi)$ is the family of periods attached to A in the usual sense. The period $p(\chi)$ determines $(\mathfrak{P}_\chi, p_\chi)$ up to isomorphism.

Since many of the results in the following chapters will be expressed in terms of the pair $(\mathfrak{P}, p)$, we would like to have a description of it that is as explicit as the description in §6 of the Taniyama group. Unfortunately, this is probably not possible since such a description would, in particular, include an explicit description of all periods of all abelian varieties with potential complex multiplication which, as (7.1) suggests, tend to be transcendental numbers. Thus the best we can hope for is an explicit characterization of the pair $(\mathfrak{P}, p)$ that does not involve CM-motives (or abelian varieties).

It is easy to describe the period torsor $\mathfrak{Q}$ attached to the category of Artin motives: $\mathfrak{Q}$ is $\operatorname{Spec}\mathbb{Q}^{\mathrm{al}}$ regarded as a principal homogeneous space for $\operatorname{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, and its canonical $\mathbb{C}$-valued point q is that defined by the given inclusion of $\mathbb{Q}^{\mathrm{al}}$ into $\mathbb{C}$. This follows from the description of $H_{\mathrm{dR}}(X)$ given in (4.1).

This suggests that we should consider the pair $(\mathfrak{P}, \varphi)$, with φ the equivariant map $\varphi : \mathfrak{P} \to \mathfrak{Q}$. Blasius has found a description of the isomorphism class of $(\mathfrak{P}, \varphi)$. Before explaining his result, we need to review a little of the theory of a Hodge-Tate modules. Write $\mathfrak{T}_\ell = \mathfrak{T} \times \operatorname{Spec}\mathbb{Q}_\ell$ and $\mathfrak{P}_\ell = \mathfrak{P} \times \operatorname{Spec}\mathbb{Q}_\ell = \mathit{Isom}(H_\ell, H_{\mathrm{dR}} \otimes \mathbb{Q}_\ell)$.

Fix a prime ℓ, and let $D_\ell = \operatorname{Gal}(\mathbb{Q}_\ell^{\mathrm{al}}/\mathbb{Q}_\ell)$. The *$\ell$-adic cyclotomic character* is the map $\chi_{\mathrm{cyc}} : D_\ell \to \mathbb{Z}_\ell^\times$ such that $\sigma(\zeta) = \zeta^{\chi_{\mathrm{cyc}}(\sigma)}$ for each root of unity ζ in $\mathbb{Q}_\ell^{\mathrm{al}}$ of ℓ-power order. The action of D_ℓ on $\mathbb{Q}_\ell^{\mathrm{al}}$ extends by continuity to the completion $\mathbb{C}_\ell$ of $\mathbb{Q}_\ell^{\mathrm{al}}$. Let V be a $\mathbb{Q}_\ell$-vector space with a continuous action of D_ℓ. We extend the action of D_ℓ on V to $\mathbb{C}_\ell \otimes V$ by the rule:

$$\sigma(c \otimes v) = \sigma c \otimes \sigma v, \qquad \sigma \in D_\ell, \quad c \in \mathbb{C}_\ell, \quad v \in V.$$

For $m \in \mathbb{Z}$, write $V\{m\}$ for the set of $v \in \mathbb{C}_\ell \otimes V$ such that

$$\sigma(v) = \chi(\sigma)^m \cdot v.$$

It is a $\mathbb{Q}_\ell$-subspace of $\mathbb{C}_\ell \otimes V$. The inclusions of the $V\{m\}$ into $\mathbb{C}_\ell \otimes V$ define a $\mathbb{C}_\ell$-linear map

$$\mathbb{C}_\ell \otimes (\oplus_{m \in \mathbb{Z}} V\{m\}) \to \mathbb{C}_\ell \otimes V,$$

which a theorem of Tate (Serre 1967) shows to be injective. When this map is an isomorphism, the D_ℓ-module V is said to be *Hodge-Tate*.

Let B_{HT} be the ring $\mathbb{C}_\ell[T, T^{-1}]$ with D_ℓ acting according to the rule $\sigma(T) = \chi(\sigma)T$. It is an immediate consequence of the definitions that $\oplus V\{m\} = (V \otimes_{\mathbb{Q}_\ell} B_{HT})^{D_\ell}$.

The D_ℓ-module $H_\ell(A)$ is known to be Hodge-Tate for all abelian varieties, and it follows that $H_\ell(M)$ is Hodge-Tate for all CM-motives over $\mathbb{Q}$. Therefore we can define a new fibre functor H'_ℓ on $\mathbf{CM}/\mathbb{Q}$ with values in $\mathbf{Vec}_{\mathbb{Q}_\ell}$ by setting

$$H'_\ell(M) = (H_\ell(M) \otimes B_{HT})^{D_\ell}.$$

Let $\mathfrak{P}'_\ell$ be the $\mathfrak{T}_\ell$-torsor $\mathit{Isom}^{\otimes}(H_\ell, H'_\ell)$. It is represented by

$$\operatorname{Spec}(\mathbb{Q}_\ell[\mathfrak{T}_\ell] \otimes B_{HT})^{D_\ell} \qquad \text{(diagonal action of } D_\ell).$$

These definitions can be extended to $\ell = \infty$ by replacing B_{HT} with $\mathbb{C}$ and D_ℓ with $D_\infty = \operatorname{Gal}(\mathbb{C}/\mathbb{R})$.

THEOREM 7.4. *(a) $\mathfrak{P}_\ell$ is (canonically) isomorphic to $\mathfrak{P}'_\ell$ for each prime ℓ (including ∞).*

(b) The isomorphisms in (a) uniquely determine the isomorphism class of $(\mathfrak{P}, \varphi)$.

PROOF: (a) Let $H_{\mathrm{Hg}}(M) = \operatorname{Gr}(H_{dR}(M))$. A Hodge cycle s on an abelian variety A has components s_ℓ in $H'_\ell(A)$ and s_{Hg} in $H_{\mathrm{Hg}}(A)$, and Blasius shows that the isomorphism of Tate-Faltings $H'_\ell(A) \to H_{\mathrm{Hg}}(A) \otimes \mathbb{Q}_\ell$ maps one component to the other, and so defines an isomorphism of fibre functors $H'_\ell \to H_{\mathrm{Hg}}$. Since there is a canonical isomorphism of fibre functors $H_{\mathrm{dR}} \otimes \mathbb{Q}_\ell \to H_{\mathrm{Hg}} \otimes \mathbb{Q}_\ell$, this shows that there is a canonical isomorphism

$$\mathit{Isom}^{\otimes}(H_\ell, H'_\ell) \approx \mathit{Isom}^{\otimes}(H_\ell, H_{\mathrm{dR}} \otimes \mathbb{Q}_\ell),$$

as required.

(b) Let $\mathfrak{S}'$ be the affine group scheme obtained from $\mathfrak{S}$ by twisting by $\mathfrak{Q}$ according to the action of $\operatorname{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ on $\mathfrak{S}$ defined in (2.5b). Thus $\mathfrak{S}'(\mathbb{Q}^{\mathrm{al}}) = \mathfrak{S}(\mathbb{Q}^{\mathrm{al}})$ with $\operatorname{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ acting through its action on *both* $\mathfrak{S}$ and $\mathbb{Q}^{\mathrm{al}}$. There is a natural action of $\mathfrak{S}'$ on $(\mathfrak{P}, \varphi)$: if $s' \in \mathfrak{S}'(\mathbb{Q}^{\mathrm{al}})$ is represented by (s, q), then s' acts on the fibre over q by multiplication by s. Moreover, for a second pair $(\mathfrak{P}', \varphi')$, $\mathit{Isom}^{\otimes}((\mathfrak{P}, \varphi), (\mathfrak{P}', \varphi'))$ is a principal homogeneous space for $\mathfrak{S}'$. Thus the set of isomorphism classes of pairs $(\mathfrak{P}', \varphi')$ is a principal homogeneous space for $H^1(\mathbb{Q}, \mathfrak{S}')$, and Blasius shows that $H^1(\mathbb{Q}, \mathfrak{S}')$ satisfies the Hasse principle.

Theorem 7.4 satisfactorily characterizes $(\mathfrak{P}, \varphi)$. It remains to characterize the canonical complex point p. This can be done in terms of the periods of Hecke characters.

PROPOSITION 7.5. *Let p' be a point of $\mathfrak{P}(\mathbb{C})$ mapping to q. If p' maps to p_χ in $\mathfrak{P}_\chi(\mathbb{C})$ for all algebraic Hecke characters χ, then $p' = p$.*

PROOF: We can write $p' = p \cdot s$ with $s \in \mathfrak{S}(\mathbb{C})$, and the condition implies that $\chi(s) = 1$ for all characters χ of $\mathfrak{S}$.

Remark 7.6. (a) It suffices to assume that the condition in (7.5) holds for enough Hecke characters χ so that their infinity types generate $X^*(\mathfrak{S})$; for example, it suffices to take the Hecke characters arising from abelian varieties with complex multiplication. Thus the combination of (7.4) and (7.5) characterizes the periods of abelian varieties over $\mathbb{Q}$ of potential CM-type in terms of the periods of abelian varieties defined over a number field and with complex multiplication defined over that field.

(b) Blasius (1986) shows that certain products of the periods of the motives attached to Hecke characters are equal to critical values of the L-series of the Hecke character. If it could be shown that $(\mathfrak{P}, p)$ is characterized by the property in (7.4) and the critical values of Hecke L-series, this would be the characterization sought.

Notes. Theorem 7.4 is proved in Blasius (1989). The monograph (Schappacher 1988) provides a detailed introduction to the periods of motives of CM-type.

II. SHIMURA VARIETIES

In this chapter, we define Shimura varieties and state the main theorems on canonical models: every Shimura variety $\mathrm{Sh}(G, X)$ has a (unique) canonical model $\mathrm{Sh}(G, X)_E$ over its reflex field $E(G, X)$; for each $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, $\tau\mathrm{Sh}(G, X)_E$ is the canonical model over $\tau E(G, X)$ of an explicitly determined Shimura variety $\mathrm{Sh}({}^\tau G, {}^\tau X)$.

1. Connected Shimura varieties over $\mathbb{C}$.

A *bounded symmetric domain* is a bounded open connected subset D of $\mathbb{C}^m$, some m, that is symmetric in the sense that, for each point $x \in D$, there is an involutive automorphism s_x of D (the *symmetry with respect to* x) having x as an isolated fixed point. The simplest bounded symmetric domain is the open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$.

A complex manifold isomorphic to a bounded symmetric domain will be called a *symmetric Hermitian domain.* The simplest example of a symmetric Hermitian domain is the complex upper-half-plane, $H^+ = \{z \in \mathbb{C} \mid |\mathrm{Im}(z) > 0\}$. The Bergmann metric on a bounded symmetric domain provides it with a natural structure of a Hermitian manifold. Thus every symmetric Hermitian domain D has a

Hermitian structure which is invariant under all automorphisms; in particular, D is symmetric as a Hermitian manifold.

Let D be a symmetric Hermitian domain. The group $\mathrm{Aut}(D)$ of automorphisms of D (as a complex manifold) is a real semisimple Lie group with only finitely many connected components, and trivial centre. If G is a connected simple real algebraic group with trivial centre such that $D = G(\mathbb{R})^+/K$ for some maximal compact subgroup K of $G(\mathbb{R})^+$, then $\mathrm{Aut}(D) \cap G(\mathbb{R}) = G(\mathbb{R})^+$, and $G(\mathbb{R})$ has either one or two connected components.

Locally symmetric varieties. Let D be a symmetric Hermitian domain, and let G be a semisimple algebraic group over $\mathbb{Q}$ such that $D = G(\mathbb{R})^+/K$ with K a maximal compact subgroup of $G(\mathbb{R})^+$. Let Γ be an arithmetic subgroup in $G(\mathbb{Q})$, which we suppose to be torsion-free. Then $S =_{\mathrm{df}} \Gamma\backslash D$ will again be a complex manifold.

THEOREM 1.1. *The complex manifold S has a canonical structure of an algebraic variety. With this structure, every holomorphic map $V^{\mathrm{an}} \to S$ from a complex algebraic variety V (viewed as an analytic space) to S is a morphism of algebraic varieties.*

PROOF: The first statement is the theorem of Baily and Borel (1966). It can also be regarded as a special case of the more general theorem of Nadel and Tsuji (1988). The second statement is proved in Borel (1972), 3.10.

The second statement shows that the algebraic structure on S is not only canonical but is also unique. With this structure, S is called a *locally symmetric variety.*

Remark 1.2. If D is has no factors isomorphic to the unit disk, then the algebraic structure on S can be described as follows. Let Ω^1 be the sheaf of holomorphic differentials on S (regarded as a complex manifold), and let $\omega = \wedge^d\Omega^1$, $d = \dim S$. Then $A = \oplus_{n\geq 0}\Gamma(S, \omega^{\otimes n})$ is a graded ring, and there is a canonical map $S \to \mathrm{Proj}\, A$, which identifies S with an open subvariety of $\mathrm{Proj}\, A$. Since $\mathrm{Proj}\, A$ is a projective algebraic variety, this shows that S is a quasi-projective algebraic variety.

This description extends to the case where D has factors isomorphic to the unit disk provided $\Gamma(S, \omega^{\otimes n})$ is replaced with the group of sections of $\omega^{\otimes n}$ having at worst logarithmic poles along the boundary in some smooth compactification of S (see Iitaka 1982, XI, for the definitions).

Let $\bar{S}$ be the closure of S in $\operatorname{Proj} A$. Then Borel (1972) shows that $\bar{S}$ has the following property: for any nonsingular algebraic variety V containing S as an open subvariety and such that the complement of S in V has only normal crossings as singularities, there is a unique morphism $V \to \bar{S}$ whose restriction to S is the identity map. For this reason, $\bar{S}$ is called the *minimal compactification* of S (alternatively, the *Satake-Baily-Borel compactification* of S).

The axioms for a connected Shimura variety. A connected Shimura variety is a projective system of locally symmetric varieties. The datum needed to define it is a pair (G, X^+) comprising a semisimple group G over $\mathbb{Q}$ and a $G^{\mathrm{ad}}(\mathbb{R})^+$ conjugacy class X^+ of homomorphisms $\mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$ satisfying the following conditions:

(1.3.1) when composed with $\mathbb{G}^{\mathrm{ad}}_{\mathbb{R}} \to GL(\mathfrak{g})$, each h in X^+ defines a Hodge structure on $\mathfrak{g}$; this Hodge structure is required to be of type $\{(-1,1),(0,0),(1,-1)\}$;

(1.3.2) for each h in X^+, ad $h(i)$ is a Cartan involution of $G_{\mathbb{R}}$;

(1.3.3) G^{ad} has no factor defined over $\mathbb{Q}$ whose real points form a compact group.

Remark 1.4. (a) It suffices to check the conditions in (1.3.1) and (1.3.2) for a single $h \in X^+$.

(b) Axiom (1.3.1) implies that the Hodge structure on $\mathfrak{g}$ defined by h has weight zero. Hence the weight map w_h (see I.2) is trivial, and so h factors through $\mathbb{S} \to \mathbb{S}/\mathbb{G}_m$.

(c) Since $h(i)^2 = h(-1) = 1$, ad $h(i)$ is an involution of $G_{\mathbb{R}}$. To say that it is a Cartan involution means that the corresponding real form G' of G, with complex conjugation $g \mapsto h(i) \cdot \bar{g} \cdot h(i)^{-1}$, is compact. Equivalently, for every representation (V, ξ) of G, the Hodge structure $(V, \xi \circ h)$ admits a G-invariant polarization (see Deligne 1972, 2.8).

(d) Axiom (1.3.3) is included for the sake of convenience. It has the following consequence: let H be a simple factor of the simply connected covering group G^{sc} of G; then $H(\mathbb{R})$ is not compact, and so the strong approximation theorem shows that $H(\mathbb{R}) \cdot H(\mathbb{Q})$ is dense in $H(\mathbb{A})$. This implies that $H(\mathbb{Q})$ is dense in $H(\mathbb{A}_f)$. Thus $G^{\mathrm{sc}}(\mathbb{Q})$ is dense in $G^{\mathrm{sc}}(\mathbb{A}_f)$.

Example 1.5. Let $G = SL_2$, and let X^+ be the set of $PGL_2(\mathbb{R})^+$ conjugates of

$$h_0 : \mathbb{S} \to G_{\mathbb{R}}, \qquad a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Then (G, X^+) satisfies the axioms (1.3). If we write $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$, then $\mathrm{ad}(g) \circ h_0 \mapsto g \cdot i$ identifies X^+ with H^+, the complex upper-half-plane.

The complex structure on X^+. Let (G, X^+) satisfy the axioms (1.3). Fix a point $o \in X^+$, and let K_o be the subgroup of $G(\mathbb{R})^+$ fixing o. Then the action of $G(\mathbb{R})^+$ on X^+ defines a bijection

$$(*) \qquad G(\mathbb{R})^+/K_o \to X^+$$

Since K_o is fixed by ad $h_o(i)$, axiom (1.3.2) implies that it is compact; moreover

$$\mathfrak{g} = \mathfrak{k}_o + \mathfrak{p}_o, \quad \mathfrak{g} = \mathrm{Lie}\, G, \quad \mathfrak{k}_o = \mathrm{Lie}\, K_o,$$

where $\mathfrak{k}_o$ and $\mathfrak{p}_0$ are the +1 and -1 eigenspaces for ad $h(i)$ acting on $\mathfrak{g}$. When we use (*) to endow X^+ with a real analytic structure, then (*) identifies $\mathfrak{p}_o$ with $\mathrm{Tgt}_o(X^+)$. There is a unique homogeneous complex structure on X^+ such that the action of i on $\mathrm{Tgt}_o(X^+)$ corresponds to the action of $h(e^{2\pi i/8})$ on $\mathfrak{p}_o$, and relative to this structure, X^+ becomes a symmetric Hermitian domain.

Since I prefer to regard X^+ as a symmetric Hermitian domain rather than a conjugacy class of homomorphisms, I write x for a point of X^+ (thought of as a domain) and h_x for the corresponding homomorphism $\mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$; thus $h_{g \cdot x} = \mathrm{ad}(g) \circ h_x$ for $g \in G^{\mathrm{ad}}(\mathbb{R})^+$ and $x \in X^+$. Also μ_x denotes the cocharacter $z \mapsto h_{x,\mathbb{C}}(z, 1)$ attached to h_x (see I.2).

The connected Shimura variety. We now construct the connected Shimura variety associated with a pair (G, X^+). A *congruence subgroup* of $G(\mathbb{Q})$ is a subgroup of the form $\Gamma = K \cap G(\mathbb{Q})$ with K a compact open subgroup of $G(\mathbb{A}_f)$. Endow $G^{\mathrm{ad}}(\mathbb{Q})$ with the topology for which the images of the congruence subgroups in $G(\mathbb{Q})$ form a fundamental system of neighbourhoods of the identity element, and let $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$ be the completion of $G^{\mathrm{ad}}(\mathbb{Q})^+$ relative to this topology. The connected Shimura variety $\mathrm{Sh}^0(G, X^+)$ will be a scheme with a continuous right action of $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$ in the sense of §10 below.

Let $\Sigma(G)$ be the set of torsion-free arithmetic subgroups of $G^{\mathrm{ad}}(\mathbb{Q})^+$ that contain the image of a congruence subgroup of $G(\mathbb{Q})$. For $\Gamma \in \Sigma(G)$, $\Gamma \backslash X^+$ is a locally symmetric algebraic variety. The group $G^{\mathrm{ad}}(\mathbb{Q})^+$ acts on the projective system $(\Gamma \backslash X^+)_{\Gamma \in \Sigma(G)}$ as follows: for each $\Gamma \in \Sigma(G)$ and $g \in G^{\mathrm{ad}}(\mathbb{Q})^+$, g defines a map

$$\Gamma \backslash X^+ \to g^{-1}\Gamma g \backslash X^+, \qquad [x] \mapsto [g^{-1}x].$$

This map is holomorphic, and hence algebraic by (1.1). The action of $G^{\mathrm{ad}}(\mathbb{Q})^+$ on $(\Gamma\backslash X^+)_{\Gamma\in\Sigma(G)}$ extends by continuity to $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$. The *connected Shimura variety* $\mathrm{Sh}^0(G,X)$ is defined to be the projective system $(\Gamma\backslash X^+)_{\Gamma\in\Sigma(G)}$ (or its limit) together with the continuous right action of $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$ just defined.

When G is simply connected, some simplifications occur. Then $G(\mathbb{R})$ is connected, and (1.4d) shows that $G(\mathbb{Q})\cdot K = G(\mathbb{A}_f)$. For any congruence subgroup $\Gamma = G(\mathbb{Q})\cap K$ of $G(\mathbb{Q})$,

$$[x]\mapsto [x,1], \quad \Gamma\backslash X^+ \mapsto G(\mathbb{Q})\backslash X^+ \times G(\mathbb{A}_f)/K$$

is an isomorphism (on the right, $[qx,qak] = [x,a]$, for $q\in G(\mathbb{Q})$, $k\in K$).

In the limit,

$$\mathrm{Sh}^0(G,X)(\mathbb{C}) = \varprojlim \Gamma\backslash X^+ = G(\mathbb{Q})\backslash X^+ \times G(\mathbb{A}_f),$$

(apply 10.1 below). The semi-direct product $G(\mathbb{A}_f)\rtimes G^{\mathrm{ad}}(\mathbb{Q})^+$ acts on this scheme:

$$[x,a](g,q) = [q^{-1}x, ad(q^{-1})(ag)], x\in X^+, a,g\in G(\mathbb{A}_f), q\in G^{\mathrm{ad}}(\mathbb{Q})^+.$$

The homomorphism $q\mapsto (q^{-1}, \mathrm{ad}\, q)$ identifies $G(\mathbb{Q})$ with a normal subgroup $G(\mathbb{A}_f)\rtimes G^{\mathrm{ad}}(\mathbb{Q})^+$, and the quotient group $G(\mathbb{A}_f) *_{G(\mathbb{Q})} G^{\mathrm{ad}}(\mathbb{Q})^+$ continues to act on $\mathrm{Sh}^0(G,X^+)$. In this case

$$G(\mathbb{A}_f) *_{G(\mathbb{Q})} G^{\mathrm{ad}}(\mathbb{Q})^+ = G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$$

(Deligne 1979, 2.1.6.2), and the action just described agrees with that defined in the preceding paragraph.

Example 1.6. If Γ is an arithmetic subgroup of $PGL_2(\mathbb{Q})$ containing the image of a congruence subgroup in $SL_2(\mathbb{Q})$, then $\Gamma\backslash H^+$ is (by definition) an *elliptic modular curve* . Thus $\mathrm{Sh}^0(SL_2,H^+)$ is the projective system of elliptic modular curves equipped with a continuous right action of $PGL_2(\mathbb{Q})^{+\wedge}$. This is the object of study of Shimura (1971b).

Etale coverings and automorphisms of connected Shimura varieties. Connected Shimura varieties behave as though they are simply connected: a finite étale equivariant morphism from one connected Shimura variety to a second is an isomorphism (Milne 1983, 2.1). It is possible to compute the group of $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$-equivariant automorphisms of $\mathrm{Sh}^0(G, X^+)$; for example, if $G = G^{\mathrm{ad}}$, then this group is zero (ib., 2.4). The full group of (not necessarily equivariant) automorphisms of $\mathrm{Sh}^0(G, X^+)$ contains $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$ as a subgroup of finite index (Milne and Shih 1981b, 1.3).

Notes. The axioms for a connected Shimura variety are those of Deligne (1979), 2.1.8.

2. Shimura varieties over $\mathbb{C}$. For many reasons, for example, in order to have models over number fields of finite degree, it is necessary to consider nonconnected Shimura varieties. They are defined by reductive groups rather than semisimple groups. The connected Shimura varieties occur as the connected components of Shimura varieties.

The axioms for a Shimura variety. The datum needed to define a Shimura variety is a pair (G, X) comprising a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$ satisfying the following conditions:

(2.1.1) for each $x \in X$, the Hodge structure on $\mathfrak{g}$ defined by h_x is of type $\{(-1,1),(0,0),(1,-1)\}$;

(2.1.2) for each $x \in X$, $\operatorname{ad} h_x(i)$ is a Cartan involution on $G_{\mathbb{R}}^{\mathrm{ad}}$;

(2.1.3) G^{ad} has no factor defined over $\mathbb{Q}$ whose real points form a compact group;

(2.1.4) the identity component $Z(G)^0$ of the centre of $Z(G)$ of G splits over a CM-field.

Simplifications occur when (2.1.2) is replaced by a stronger axiom:

(2.1.2*) let $Z_0(G)$ be the maximal subtorus of $Z(G)$ split over $\mathbb{Q}$; then $\operatorname{ad} h_x(i)$ is a Cartan involution on $G/Z_0(G)$.

We say that (G, X) satisfies (2.1) when it satisfies (2.1.1) - (2.1.4); when it also satisfies (2.1.2*), we say that it satisfies (2.1*).

Remark 2.2. (a) Again it suffices to check (2.1.1) and (2.1.2) for a single $x \in X$.

(b) Let X^+ be a connected component of X, and for each $x \in X^+$, let h'_x be the composite of h_x with $G_{\mathbb{R}} \to G_{\mathbb{R}}^{\mathrm{ad}}$. Then $x \mapsto h'_x$ identifies X^+ with a $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}^{\mathrm{ad}}$, which satisfies the axioms (1.3). Therefore X^+ acquires from §1 a natural structure of a symmetric Hermitian domain, and so X is a finite disjoint union of symmetric Hermitian domains (indexed by $G(\mathbb{R})/G(\mathbb{R})_+$).

(c) Axiom (2.1.1) implies that the Hodge structure on $\mathfrak{g}$ defined by $\mathrm{ad} \circ h_x$ has weight zero. Hence the weight map w_x is central, and so it is independent of x — we write it w_X.

(d) Axiom (2.1.4) is not in Deligne's list of axioms (Deligne (1979), 2.1.1), but it is harmless to impose it since, in practice, all examples satisfy it, and it allows some simplifications; for example, it implies that w_X is defined over a totally real field.

(e) Axiom (2.1.2*) is very restrictive; it excludes many important Shimura varieties, for example, all Hilbert modular varieties of dimension greater than one.

Example 2.3. Let V be a vector space of dimension 2 over $\mathbb{Q}$. Let $G = GL(V)$, and let X be the set of complex structures on $V \otimes \mathbb{R}$. With each $x \in X$ we associate the homomorphism $h_x : \mathbb{S} \to G_{\mathbb{R}}$ such that $h_x(z)$ acts on $V \otimes \mathbb{R}$ as z for all $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$. Then $x \mapsto h_x$ identifies X with a $G(\mathbb{R})$-conjugacy class of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$, and the pair (G, X) satisfies the axioms (2.1). The choice of a basis for V identifies G with GL_2 and X with $\mathbb{C} - \mathbb{R} = \{z \in \mathbb{C} \mid \Im(z) \neq 0\}$, the union of the upper and lower half-planes.

The Shimura variety. Let (G, X) satisfy the conditions (2.1). For K a compact open subgroup in $G(\mathbb{A}_f)$, consider the double coset space

$$\mathrm{Sh}_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

where

$$q(x, a)k = (qx, qak), \quad q \in G(\mathbb{Q}), x \in X,\ a \in G(\mathbb{A}_f),\ k \in K.$$

Let $\mathcal{C}$ be a set of representatives for the finite set $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$, and, for each $g \in \mathcal{C}$, let Γ_g be the image in $G^{\mathrm{ad}}(\mathbb{R})^+$ of the subgroup $\Gamma'_g = gKg^{-1} \cap G(\mathbb{Q})_+$ of $G(\mathbb{Q})_+$. Then

$$\mathrm{Sh}_K(G, X) = \cup \Gamma_g\backslash X^+ \quad \text{(disjoint union over } g \in \mathcal{C})$$

for any connected component X^+ of X. When K is sufficiently small, Γ_g will be torsion-free, and we conclude from (1.1) that $\mathrm{Sh}_K(G, X)$ will then be a finite disjoint union of locally symmetric varieties. It therefore has a unique structure of an algebraic variety. Let

$$\mathrm{Sh}(G, X) = \varprojlim \mathrm{Sh}_K(G, X).$$

This is a scheme over $\mathbb{C}$ whose complex points are

$$\mathrm{Sh}(G,X)=G(\mathbb{Q})\backslash X\times G(\mathbb{A}_f)/Z(\mathbb{Q})^-,$$

where $Z(\mathbb{Q})^-$ is the closure of $Z(\mathbb{Q})$ in $Z(\mathbb{A}_f)$ (to prove this, apply (10.1) below with $E=G(\mathbb{Q})\backslash X\times G(\mathbb{A}_f)/Z(\mathbb{Q})^-$). When the maximal $\mathbb{R}$-split subtorus of $Z(G)$ is $\mathbb{Q}$-split, $Z(\mathbb{Q})$ is closed in $Z(\mathbb{A}_f)$, and so

$$\mathrm{Sh}(G,X)=G(\mathbb{Q})\backslash X\times G(\mathbb{A}_f).$$

There is a continuous action of $G(\mathbb{A}_f)$ on $\mathrm{Sh}(G,X)$, given by

$$[x,a]g=[x,ag],\quad x\in X,\ a\in G(\mathbb{A}_f),\ g\in G(\mathbb{A}_f).$$

The scheme $\mathrm{Sh}(G,X)$ together with this continuous action of $G(\mathbb{A}_f)$ is called the *Shimura variety* defined by (G,X). We write (g) or $\mathcal{T}(g)$ for the operation of $g\in G(\mathbb{A}_f)$ on $\mathrm{Sh}(G,X)$ — it is often called the *Hecke operator* defined by g.

Example 2.4. (a) A *symplectic space* over $\mathbb{Q}$ is a vector space V over $\mathbb{Q}$ together with a nondegenerate skew-symmetric form ψ on V. The group $G=GSp(V,\psi)$ of symplectic similitudes of (V,ψ) has rational points

$$G(\mathbb{Q})=\{\alpha\in GL(V)\mid\exists q\in\mathbb{Q}^\times \text{s.t.}\psi(\alpha v,\alpha w)=q\psi(v,w),\forall v,w\in V\}.$$

Let $S^\pm$ be the set of all Hodge structures of type $\{(-1,0),(0,-1)\}$ on V for which $\pm 2\pi i\psi$ is a polarization. Then $S^\pm$ is a $G(\mathbb{R})$-conjugacy class of homomorphisms $\mathbb{S}\to G_{\mathbb{R}}$, and the pair $(G,S^\pm)$ satisfies the conditions (2.1). The space $S^\pm$, regarded as a disjoint union of two Hermitian symmetric domain, is the *Siegel double space* , and the variety $\mathrm{Sh}(G,S^\pm)$ is the *Siegel modular variety* .

(b) Let F be a totally real number field, and let $G=GL_{2,F}$, so that $G(\mathbb{R})=\prod_{\mathrm{Hom}(F,\mathbb{R})}GL_2(\mathbb{R})$. Let X be the set of $G(\mathbb{R})$-conjugates of

$$h_0:\mathbb{S}\to G_{\mathbb{R}},\quad a+ib\mapsto\left(\begin{pmatrix}a&-b\\b&a\end{pmatrix},\begin{pmatrix}a&-b\\b&a\end{pmatrix},\dots,\begin{pmatrix}a&-b\\b&a\end{pmatrix}\right).$$

Then X is a product of $[E:\mathbb{Q}]$ copies of $\mathbb{C}-\mathbb{R}$, and (G,X) satisfies the axioms (2.1). The variety $\mathrm{Sh}(G,X)$ is the *Hilbert modular variety*.

Remark 2.5. The semi-direct product $G(\mathbb{A}_f)/Z(\mathbb{Q})^- \rtimes G^{\mathrm{ad}}(\mathbb{Q})^+$ acts on $\mathrm{Sh}(G,X)$. Moreover, the quotient

$$\mathcal{G}(G) =_{\mathrm{df}} (G(\mathbb{A}_f)/Z(\mathbb{Q})^-) *_{G(\mathbb{Q})_+/Z(\mathbb{Q})} G^{\mathrm{ad}}(\mathbb{Q})^+$$

of this group by its normal subgroup

$$\{(q^{-1}, \mathrm{ad}\, q) \mid q \in G(\mathbb{Q})_+/Z(\mathbb{Q})\}$$

continues to act. The Shimura variety $\mathrm{Sh}(G,X)$ is a scheme with a continuous action of $\mathcal{G}(G)$ in the sense of §10 below.

The reflex field. The reflex field is the natural field of definition of the Shimura variety. It is defined purely in terms of G and X.

For any field k of characteristic zero, let $\mathcal{M}\ (k)$ be the set of $G(k)$-conjugacy classes of homomorphisms $\mathbb{G}_m \to G_k$. The map $\mathcal{M}(k_1) \to \mathcal{M}(k_2)$ defined by an inclusion $k_1 \hookrightarrow k_2$ of algebraically closed fields is bijective. In particular, $\mathcal{M}(\mathbb{Q}^{\mathrm{al}}) \approx \mathcal{M}(\mathbb{C})$.

The cocharacters μ_x for x in X lie in a single class $M_X \in \mathcal{M}(\mathbb{C})$, which we can regard as an element of $\mathcal{M}(\mathbb{Q}^{\mathrm{al}})$. The *reflex field* $E(G,X)$ is the fixed field of the subgroup $\{\sigma \mid \sigma M_X = M_X\}$ of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$; it is therefore the field of definition of the conjugacy class M_X. With our axiom (2.1.4), $E(G,X)$ will be contained in a CM-field (see Deligne 1971c, 3.8), which means that it is either a CM-field or a totally real field.

Special points. A point $x \in X$ is *special* if there is a maximal $\mathbb{Q}$-rational torus $T \subset G$ such that h_x factors through $T_{\mathbb{R}}$ (equivalently, $T(\mathbb{R})$ fixes x). Then

$$\mu_x^{\mathrm{ad}} =_{\mathrm{df}} (\mathbb{G}_m \xrightarrow{\mu_x} T \to T/Z(G) \subset G^{\mathrm{ad}})$$

satisfies the Serre condition, and so there is a unique homomorphism $\rho_x^{\mathrm{ad}} : \mathfrak{S} \to G^{\mathrm{ad}}$ such that $\mu_{\mathrm{can}} \circ (\rho_x^{\mathrm{ad}})_{\mathbb{C}} = \mu_x^{\mathrm{ad}}$ (see I.2.4b). There always exist many special points in X (Deligne 1971c, 5.1).

When μ_x itself satisfies the Serre condition, we call x a *CM-point.* In this case there exists a unique $\mathbb{Q}$-rational homomorphism $\rho_x : \mathfrak{S} \to G$ such that $\mu_{\mathrm{can}} \circ (\rho_x)_{\mathbb{C}} = \mu_x$. A $\mathbb{Q}$-linear representation (V,ξ) of G attaches a CM-motive over $\mathbb{Q}^{\mathrm{al}}$ to each CM-point x, namely, that corresponding to the representation $(V, \xi \circ \rho_x)$ of $\mathfrak{S}$ (see I.4). The existence of a single CM-point implies that the weight w_X is defined over $\mathbb{Q}$, and conversely, if w_X is defined over $\mathbb{Q}$, then every special point is CM (under our axiom (2.1.4); see Milne (1988), A.3).

A pair (T, x) comprising a point x of X (necessarily special) and a maximal torus $T \subset G$ such that h_x factors through $T_{\mathbb{R}}$ will be called a *special pair* in (G, X). When x is a CM-point, we refer to a CM-*pair*.

A point $[x, g]$ of $\mathrm{Sh}(G, X)$ is said to be *special* (or CM) if x is special (or CM) in X. There is always a special point in X, and for any special point x, $[x, 1] \cdot G(\mathbb{A}_f)$ is dense in $\mathrm{Sh}(G, X)$ for the Zariski topology (Deligne 1971c, 5.1).

Shimura varieties defined by tori. Let T be a torus over $\mathbb{Q}$ split by a CM-field. A pair (T, x), $h_x : \mathbb{S} \to T_{\mathbb{R}}$, automatically satisfies the axioms (2.1). The associated Shimura variety

$$\mathrm{Sh}(T, x) = \varprojlim T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K = T(\mathbb{A}_f)/T(\mathbb{Q})^-$$

has dimension zero. The reflex field $E(T, x)$ of (T, x) is the field of definition of μ_x.

For example, let E be a CM-field and Φ a CM-type for E. Then (T^E, h_Φ) defines a Shimura variety whose reflex field is $E^*(\Phi)$, the reflex field of (E, Φ). (Notations as in I.2.6.)

Morphisms of Shimura varieties. Let (G, X) and (G', X') be pairs satisfying (2.1). By a *morphism* $f : (G, X) \to (G', X')$, we mean a homomorphism $f : G \to G'$ mapping X into X'. Such an f defines a morphism of schemes

$$\mathrm{Sh}(f) : \mathrm{Sh}(G, X) \to \mathrm{Sh}(G', X'), \quad [x, a] \mapsto [f(x), f(a)]$$

which is equivariant for $f : G(\mathbb{A}_f) \to G'(\mathbb{A}_f)$, that is,

$$\mathrm{Sh}(f) \circ \mathcal{T}(g) = \mathcal{T}(f(g)) \circ \mathrm{Sh}(f), \quad \text{for } g \in G(\mathbb{A}_f).$$

If $f : G \to G'$ is a closed immersion, then so also is $\mathrm{Sh}(f)$ (Deligne 1971c, 1.15).

PROPOSITION 2.6. *Let (G, X) and (G', X') be two pairs satisfying (2.1), and suppose given*

(i) a morphism $f_1 : (G, X) \to (G', X')$;

(ii) a continuous homomorphism $f_2 : G(\mathbb{A}_f) \to G'(\mathbb{A}_f)$;

(iii) an element $a \in G_1(\mathbb{A}_f)$ such that $f_1 \circ \operatorname{ad} a^{-1} = f_2$.

Then the morphism $\varphi =_{\mathrm{df}} \mathrm{Sh}(f_1) \circ \mathcal{T}(a) : \mathrm{Sh}(G_1, X_1) \to \mathrm{Sh}(G_2, X_2)$ *maps* $[x, a^{-1}]$ *to* $[f_1(x), 1]$ *for all* $x \in X_1$*, and is equivariant:*

$$\varphi \circ \mathcal{T}(g) = \mathcal{T}(f_2(g)) \circ \varphi \text{ for all } g \in G_1(\mathbb{A}_f).$$

Moreover, φ *is unchanged when* f_1 *is replaced with* $f_1 \circ \mathrm{ad}\, q$*,* $q \in G(\mathbb{Q})$*, and* a *with* aq*.*

PROOF: Straightforward.

The relation between connected and nonconnected Shimura varieties. Let X^+ be a connected component of X, and let $\mathrm{Sh}(G, X)^0$ be the connected component of $\mathrm{Sh}(G, X)$ containing the image of X^+. As we observed in (2.2b), X^+ can be identified with a $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class of homomorphisms $\mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$. It is an important observation of Deligne that $\mathrm{Sh}(G, X)^0$ can be described solely in terms of G^{der} and X^+; in particular, it is independent of the centre of G (except for $Z(G) \cap G^{\mathrm{der}}$).

PROPOSITION 2.7. *Let* (G, X) *be a pair satisfying (2.1), and let* X^+ *be a connected component of* X*. When* X^+ *is regarded as a conjugacy class of maps* $\mathbb{S} \to G^{\mathrm{ad}}(\mathbb{R})^+$*, the pair* (G^{der}, X^+) *satisfies the axioms (1.3), and*

$$[x] \mapsto [x, 1] : \mathrm{Sh}^0(G, X^+) \to \mathrm{Sh}(G, X)$$

defines an equivariant isomorphism of $\mathrm{Sh}^0(G, X^+)$ *onto* $\mathrm{Sh}(G, X)^0$. *The stabilizer of* $\mathrm{Sh}^0(G, X)$ *in* $\mathcal{G}(G)$ *is* $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$.

PROOF: Deligne (1979), 2.1.16.

In the language of §10 below, the proposition says that $\mathrm{Sh}(G, X)$ is obtained from $\mathrm{Sh}^0(G^{\mathrm{der}}, X^+)$ by induction from $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$ to $\mathcal{G}(G)$. This result will enable us to relate statements about connected Shimura varieties to statements about nonconnected Shimura varieties. To this end, the following result, which shows that each connected Shimura variety occurs as a connected component of a particularly good Shimura variety, is useful.

PROPOSITION 2.8. *For any pair* (G, X^+) *defining a connected Shimura variety, there is a pair* (G_1, X_1) *defining a Shimura variety and such that:*

(a) $(G_1^{\mathrm{der}}, X_1^+) = (G, X^+)$;

(b) the weight w_{X_1} *is defined over* $\mathbb{Q}$.

Moreover, G_1 *can be chosen so that either:*

(c) $H^1(k, Z(G_1)) = 0$ *for all fields* $k \supset \mathbb{Q}$*; or*

(d) $\operatorname{ad} h(i)$ *is a Cartan involution on* $G_1/w_{X_1}(\mathbb{G}_m)$ *(hence (2.1.2*) holds).*

PROOF: See the Appendix to Milne (1988).

The minimal compactification of $\mathrm{Sh}(G, X)$**.** Assume that G^{ad} has no factors of dimension 3, and let

$$A = \oplus_{n \geq 0} \Gamma(\mathrm{Sh}(G, X), \omega^{\otimes n}), \quad \omega = \Lambda^d \Omega^1, \quad d = \dim X.$$

There is a canonical inclusion $\mathrm{Sh}(G, X) \to \operatorname{Proj} A$, the closure of whose image, $\overline{\mathrm{Sh}(G, X)}$, is called the *minimal (or Satake-Baily-Borel) compactification* of $\mathrm{Sh}(G, X)$. When G^{ad} has factors of dimension 3, we must replace $\Gamma(S, \omega^{\otimes n})$ with the group of sections having at worst logarithmic singularities along the boundary of some smooth compactification of $\mathrm{Sh}(G, X)$ (cf. 1.2).

Automorphisms of Shimura varieties. It is possible to use the results in §1 on automorphisms of connected Shimura varieties to compute the group of $G(\mathbb{A}_f)$-equivariant automorphisms of a Shimura variety. Clearly the Hecke operator $\mathcal{T}(g)$ associated with any $g \in Z(\mathbb{A}_f)$ is such an automorphism of $\mathrm{Sh}(G, X)$, and conversely one can show that when $Z(G)$ satisfies the Hasse principle for finite primes, that is, $H^1(\mathbb{Q}, Z(G)) \hookrightarrow \prod_{\text{finite primes}} H^1(\mathbb{Q}_\ell, Z(G))$, then all $G(\mathbb{A}_f)$-automorphisms of $\mathrm{Sh}(G, X)$ are of this form. Thus, in this case,

$$\mathrm{Aut}_{G(\mathbb{A}_f)} \mathrm{Sh}(G, X) = Z(\mathbb{A}_f)/Z(\mathbb{Q})^-.$$

See Milne (1983), 2.7.

Notes. The axioms for a Shimura variety were introduced in Deligne (1971c) and, in slightly revised form, in Deligne (1979). They were suggested by the work of Shimura. This section summarizes parts of the two articles of Deligne.

3. Shimura varieties as moduli varieties for motives.

In this section, we explain how the choice of a representation $\xi : G \to GL(V)$, V a $\mathbb{Q}$-vector space, endows $\mathrm{Sh}(G, X)$ with all the additional structure that a family of motives over $\mathrm{Sh}(G, X)$ would give. This suggests that, under some restrictions on (G, X), $\mathrm{Sh}(G, X)$ should be a fine moduli space for motives.

Review of local systems and flat vector bundles. Let S be an algebraic variety over k, and let $\mathcal{V}$ be a vector bundle on S. A *connection* on $\mathcal{V}$ is a k-linear homomorphism

$$\nabla : \mathcal{V} \to \Omega^1_S \otimes \mathcal{V} \qquad (\mathcal{V} \text{ regarded as a sheaf })$$

satisfying the Leibniz identity,

$$\nabla(fv) = df \cdot v + f \cdot \nabla v$$

for all local sections f of $\mathcal{O}_S$ and v of $\mathcal{V}$. A vector field Z on S defines a mapping $\nabla_Z : \mathcal{V} \to \mathcal{V}$ by the rule: for a section v of $\mathcal{V}$ on an open subset U of S,

$$\nabla_Z(v) = \langle \nabla v, Z \rangle \in \Gamma(U, \mathcal{V}).$$

A connection is said to be *flat* (or *integrable*) if its curvature tensor is zero, that is,

$$\nabla_Y \cdot \nabla_Z - \nabla_Z \cdot \nabla_Y = \nabla_{[Y,Z]}, \quad \text{all } Y \text{ and } Z.$$

A local section v of $\mathcal{V}$ is said to be *horizontal* for ∇ if $\nabla v = 0$. A vector bundle with a flat connection can be regarded as a $\mathcal{D}$-module, where $\mathcal{D}$ is the ring of differential operators — see Borel et al. (1987), Chapter VI.

These definition carry over *mutatis mutandis* to a complex manifold S. Let $\pi_1(S, s)$ be the fundamental group of S regarded as the group of covering transformations of the universal covering space $\tilde{S}$ of S (acting on the right). A complex representation $\xi : \pi_1(S, s) \to GL(V)$ defines a vector bundle on S

$$\mathcal{V}(\xi) = \tilde{S} \times V/\sim, \quad (s\gamma, v) = (s, \gamma v),\ s \in \tilde{S},\ \gamma \in \pi_1(S, s),\ v \in V,$$

having a canonical flat connection $\nabla(\xi)$. Conversely, if $\mathcal{V}$ is a vector bundle on S with a flat connection ∇, then $V =_{\mathrm{df}} \mathcal{V}^{\nabla}$ is a local system of $\mathbb{C}$-vector spaces on S, and for any such system, there is an natural representation of $\pi_1(S, s)$ on the stalk V_s of V at $s \in S$.

We refer to Borel et al. (1987), Chapter IV, for the notion of a flat connection being *regular at infinity.*

PROPOSITION 3.1. *Let S be a complex manifold. The above constructions define equivalences between:*

(a) the category of vector bundles with flat connection $(\mathcal{V},\nabla)$ on S;

(b) the category of local systems of $\mathbb{C}$*-vector spaces;*

(c) the category of complex representations of $\pi_1(S,s)$.

When X *is a smooth algebraic variety, the functor* $(\mathcal{V},\nabla)\mapsto(\mathcal{V}^{an},\nabla^{an})$ *is an equivalence from the category of algebraic vector bundles with a flat connection regular at infinity to that of analytic vector bundles with a flat connection.*

PROOF: Except for the last statement, this is a standard result. The last statement can be found in (Deligne 1970) and (Borel et al. 1987, Chapter IV).

Variations of Hodge structures. A *variation of Hodge structures* on a complex manifold S is a local system of $\mathbb{Q}$-vector spaces V on S together with a continuously varying family of Hodge structures on the stalks V_s of V such that

(a) the Hodge filtration on $(\mathbb{C}\otimes_{\mathbb{Q}}V)_s$ varies holomorphically with s, that is, it defines a filtration of the vector bundle $\mathcal{V}=_{\mathrm{df}}\mathcal{O}_S\otimes_{\mathbb{Q}}V$;

(b) (axiom of transversality): $\nabla(F^p\mathcal{V})\subset\Omega^1_S\otimes F^{p-1}\mathcal{V}$.

When $\mathbb{Q}$ is replaced by $k\subset\mathbb{R}$, we speak of a variation of Hodge k-structures. All families of Hodge structures arising naturally in algebraic geometry are variations of Hodge structures.

X as a parameter space for Hodge structures. As a first step to realizing $\mathrm{Sh}(G,X)$ as a moduli variety for motives, we show how to realize X as a parameter space for Hodge structures; in fact, the axioms (2.1) are virtually forced on us by our wish that this be so.

Let G be a connected algebraic group over $\mathbb{R}$, and let X^+ be a connected component of the space of homomorphisms $\mathbb{S}\to G_{\mathbb{R}}$. Then X^+ is a $G(\mathbb{R})^+$-conjugacy class of homomorphisms. Choose a faithful representation (V,ξ) of G. For each $x\in X^+$, we obtain a real Hodge structure $\xi\circ h_x$ on V. We assume that the corresponding weight gradation is independent of x (equivalently, $\xi\circ h_x(\mathbb{R}^\times)$ is contained in the centre of $G(\mathbb{R})^+$ for all x).

PROPOSITION 3.2. *Let* $V(\xi)$ *be the constant sheaf of* $\mathbb{R}$*-vector spaces on* X^+ *defined by* V.

(a) There is a unique complex structure on X^+ *such that the Hodge filtrations on the stalks of* $\mathbb{C}\otimes V(\xi)$ *vary holomorphically.*

(b) The Hodge structures $\xi\circ h_x$ *make* $V(\xi)$ *into a variation of real Hodge structures if and only if the Hodge structure on* $\mathfrak{g}$ *defined by* h_x *is of type* $\{(-1,1),(0,0),(1,-1)\}$ *for all* $x\in X^+$.

(c) Let G_1 be the smallest algebraic subgroup of G through which all the h_x, $x \in X^+$, factor, and let V_n be the component of V of weight n. There exists a bilinear form $\psi : V_n \otimes V_n \to \mathbb{R}(-n)$ that is a polarization of $(V_n, \xi \circ h_x)$ for all $x \in X^+$ if and only if G_1 is reductive and $\mathrm{ad}\, h_x(i)$ is a Cartan involution on G_1^{ad}, all x.

PROOF: This is proved in Deligne (1979), 1.1.14. We merely note that the Hodge filtrations on the stalks of $\mathbb{C} \otimes V(\xi)$ define a map from X into a Grassman manifold, and (a) is equivalent to this map being holomorphic. Moreover, that if Z is a vector field on X corresponding to an element of $F_x^r(\mathrm{Lie}\, G)$ then $\nabla_Z(F^s \mathcal{V}_x) \subset F^{r+s}\mathcal{V}_x$; the condition implies that $\mathrm{Lie}\, G = F_x^{-1}(\mathrm{Lie}\, G)$. Finally, the result noted in (1.4c) implies the existence of ψ.

Now assume that (G, X) is a pair satisfying (2.1). The structure on X that we defined in §2 is the unique complex structure such that every real representation (V, ξ) of G defines a variation of real polarizable Hodge structures on X. If the weight w_X is defined over $\mathbb{Q}$, then every rational representation (V, ξ) of G defines a rational polarizable variation of Hodge structures on X. We can extend $V(\xi)$ to $X \times G(\mathbb{A}_f)$, and when (2.1.2*) holds we can pass to the quotient to obtain a polarizable variation of Hodge structures (rational or real) on $\mathrm{Sh}(G, X)$. In the rational case, this variation of Hodge structures is a candidate to be the family of Betti cohomology groups of a family of motives over $\mathrm{Sh}(G, X)$.

Local systems of $\mathbb{Q}_\ell$-adic vector spaces. Let S be a scheme. By a *local system of $\mathbb{Q}_\ell$-vector spaces* on S_{et} I mean a twisted-constant constructible (or smooth) $\mathbb{Q}_\ell$-sheaf; see for example Milne (1980), p165. When S is connected and s is a geometric point of S, the map $V \mapsto V_s$ (stalk of V at s) defines an equivalence from the category of local systems of $\mathbb{Q}_\ell$-vector spaces on S to that of continuous representations of $\pi_1^{\mathrm{et}}(S, s)$ on $\mathbb{Q}_\ell$-vector spaces. More generally, if $X \to S$ is a Galois covering of S with Galois group $\mathcal{G}$ (see 10.2), then

$$V \mapsto \varinjlim V(X^{\mathcal{H}}) \quad \text{(limit over the open subgroups } \mathcal{H} \text{ of } \mathcal{G}\text{)},$$

defines an equivalence from the category of local systems of $\mathbb{Q}_\ell$-vector spaces on S whose pull-back to X is constant to that of continuous representations of $\mathcal{G}$.

Now take S to be a smooth connected variety over $\mathbb{C}$, and let $s \in S(\mathbb{C})$. In this case, s is also a geometric point of S, and $\pi_1^{\mathrm{et}}(S, s)$

is the profinite completion of $\pi_1(S,s)$. A local system of $\mathbb{Q}$-vector spaces V on $S(\mathbb{C})$ defines a representation $\xi : \pi_1(S,s) \to GL(V_s)$, which extends to a representation $\xi_\ell : \pi_1^{\text{et}}(S,s) \to GL(V_s \otimes \mathbb{Q}_\ell)$ if and only if it is continuous relative to the ℓ-adic topology on V and the profinite topology on $\pi_1(S,s)$. In this case, we abuse notation, and write $V \otimes \mathbb{Q}_\ell$ for the local system of $\mathbb{Q}_\ell$-vector spaces on S_{et} associated with ξ_ℓ.

The systems attached to a rational representation of G.

PROPOSITION 3.3. *Assume that (G,X) satisfies (2.1*). A representation (V,ξ) of G defines (in a natural way):*
(a) a local system of $\mathbb{Q}$-vector spaces $V(\xi)$ on $\mathrm{Sh}(G,X)$;
(b) a local system of $\mathbb{Q}_\ell$-vector spaces $V_\ell(\xi)$ on $\mathrm{Sh}(G,X)_{\text{et}}$, each ℓ;
(c) a vector bundle $\mathcal{V}(\xi)$ on $\mathrm{Sh}(G,X)$ together with a (regular) flat connection $\nabla(\xi)$.
These are related by canonical isomorphisms:
(i) $V(\xi) \otimes \mathbb{Q}_\ell \to V_\ell(\xi)$;
(ii) $V(\xi) \otimes \mathbb{C} \to \mathcal{V}(\xi)^{\nabla(\xi)}$.
When the weight w_X is defined over $\mathbb{Q}$, the maps $\xi \circ h_x$ define on $V(\xi)$ the structure of a variation of polarizable Hodge structures.

PROOF: Let K be compact open subgroup of $G(\mathbb{A}_f)$. Then (see §2) $\mathrm{Sh}_K(G,X)$ is a finite union $\cup \Gamma_g \backslash X^+$, where Γ_g is the image of $\Gamma'_g = gKg^{-1} \cap G(\mathbb{Q})_+$ in $G^{\text{ad}}(\mathbb{Q})_+$. When K is sufficiently small, Γ_g will be the fundamental group of $\Gamma_g \backslash X^+$. The condition (2.1.2*) implies that $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_f)$, and so we can take K to be sufficiently small so that $K \cap Z(\mathbb{Q}) = \{1\}$. Since the kernel of $\Gamma'_g \to \Gamma_g$ is contained in $Z(\mathbb{Q})$, this shows that we can assume that $\Gamma'_g = \Gamma_g$. Now each of $V(\xi)$ and $(V(\xi), \nabla(\xi))$ is defined on $\Gamma_g \backslash X^+$ by the restriction of ξ to Γ'_g. The sheaf $V_\ell(\xi)$ can be defined to be $V(\xi) \otimes \mathbb{Q}_\ell$ or, better, we can proceed as follows. The above discussion shows that when K is sufficiently small, Γ'_g will act without fixed points on X^+. Under the same hypothesis, K will act without fixed points on $\mathrm{Sh}(G,X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)$. Then $\mathrm{Sh}(G,X)$ will be a Galois covering of $\mathrm{Sh}_K(G,X)$, and we can take $\mathcal{V}_\ell(\xi)$ to be sheaf associated with the representation of K on $V \otimes \mathbb{Q}_\ell$ defined by ξ.

The motives attached to the points of $\mathrm{Sh}(G,X)$. Our discussion in this and the next subsection is predicated on the assumption of (I.3.2), so that there is a theory of motives over any field of characteristic zero, and the Betti fibre functor $\mathbf{Mot}/\mathbb{C} \to \mathbf{Hdg}_{\mathbb{Q}}$ is fully faithful (see I.4). Let (G, X) be a pair satisfying (2.1*), and assume that w_X is defined over $\mathbb{Q}$. To simplify the discussion, we assume there is a homomorphism $t : G \to \mathbb{G}_m$ such that $t \circ w_X(z) = z^{-2}$. Fix a faithful representation (V, ξ) of G.

HOPE 3.4. *Each $(V, \xi \circ h_x)$ is the rational Hodge structure attached to a motive M_x over $\mathbb{C}$ (uniquely determined, because of our assumption of I.3.2).*

As we noted in §2, when x is a CM-point we know that M_x exists, and it is a motive of CM-type. Let $\mathfrak{t} = (t_\alpha)_{\alpha \in I}$ be a family of tensors for V such that G is the subgroup of $GL(V) \times \mathbb{G}_m$ fixing the t_α.

Consider the set of triples $(M, \mathfrak{s}, \eta)$ consisting of a motive M over $\mathbb{C}$, a family $\mathfrak{s} = (s_\alpha)_{\alpha \in I}$ of Hodge cycles on M, and an isomorphism $\eta : V(\mathbb{A}_f) \to H_f(M)$ such that:

(3.5a) there exists an isomorphism $i : H_B(M) \to V$ mapping each s_α to t_α and such that $(z \mapsto i \circ h_M(z) \circ i^{-1}) \in X$;

(3.5b) η maps each s_α to t_α.

An isomorphism from one such triple $(M, \mathfrak{s}, \eta)$ to a second $(M', \mathfrak{s}', \eta')$ is an isomorphism $\gamma : M \to M'$ sending each s_α to s'_α and such that $\gamma \circ \eta = \eta'$. Write $\mathcal{M}(G, X, \xi)$ for the set of isomorphism classes of such triples.

PROPOSITION 3.6. *Under the above assumptions, there is a canonical bijection*

$$\Phi_\xi : \mathcal{M}(G, X, \xi) \to \mathrm{Sh}(G, X).$$

PROOF: Given $(M, \mathfrak{s}, \eta)$, choose an isomorphism $i : H_B(M) \to V$ as in (3.5a), and let $x \in X$ be such that $h_x(z) = i \circ h_M(z) \circ i^{-1}$. Because $i_{\mathbb{A}_f} \circ \eta : V(\mathbb{A}_f) \to V(\mathbb{A}_f)$ preserves Hodge cycles, it is multiplication by $\xi(a)$, some $a \in G(\mathbb{A}_f)$. The map i is uniquely determined up to an element of $G(\mathbb{Q})$, and so the class of (x, a) in $\mathrm{Sh}(G, X)$ is well-defined: we set $\Phi_\xi(M, \mathfrak{s}, \eta) = [x, a]$. Conversely, given $(x, a) \in X \times G(\mathbb{A}_f)$, let M_x be the motive determined by (3.4), and define t_α to be s_α and η to be multiplication by $\xi(a)$.

Remark 3.7. It is possible to recover (G, X) from the triple $(M, \mathfrak{s}, \eta)$ attached to a single point of $\mathrm{Sh}(G, X)$: by definition G is the subgroup of $GL(H_B(M)) \times \mathbb{G}_m$ fixing the s_α; because s_α is a Hodge cycle, $h_M(\mathbb{S})$ fixes it, and so h_M factors through $G_{\mathbb{R}}$; X is the $G(\mathbb{R})$-conjugacy class of h_M.

Families of motives. We define a family of motives over a scheme S to be a motive over the generic point "with good reduction everywhere".

Definition 3.8. Let S be a smooth connected variety over $\mathbb{C}$ with generic point η, and let $\bar{\eta}$ be a geometric point lying over η. A *motive* $\mathcal{M}$ over S is a motive M_η over $\mathbb{C}(\eta)$ such that the action of $\mathrm{Gal}(\mathbb{C}(\eta)^{\mathrm{al}}/\mathbb{C}(\eta))$ on $H_\ell(\mathcal{M}_\eta)$ factors through $\pi_1^{\mathrm{et}}(S,\bar{\eta})$, all ℓ.

Write $\mathcal{H}_\ell(\mathcal{M})$ for the local system of $\mathbb{Q}_\ell$-vector spaces on S_{et} defined by the representation of $\pi_1^{\mathrm{et}}(S,\bar{\eta})$ on $H_\ell(M_\eta)$. Let S_0 be a model of S over a subfield k_0 of $\mathbb{C}$ of finite transcendence degree over $\mathbb{Q}$, and let η_0 be the generic point of S_0; assume k_0 is sufficiently large that M_η has a model M_0 over η_0. For any sufficiently general closed point t of S, there will be a k-morphism $\eta_0 \to S$ with image t, and M_0 will define a motive M_t over t. There is a local system of $\mathbb{Q}$-vector spaces $\mathcal{H}_B(\mathcal{M})$ on S such that $\mathcal{H}_B(\mathcal{M})_t = H_B(M_t) \subset \mathcal{H}_\ell(\mathcal{M})_t$ for every such t. From (3.1) we then obtain a pair $(\mathcal{H}_{\mathrm{dR}}(\mathcal{M}),\nabla)$ such that $\mathcal{H}_{\mathrm{dR}}(\mathcal{M})^\nabla = \mathbb{C}\otimes\mathcal{H}_B(\mathcal{M})$.

A motive on a nonconnected smooth scheme S over $\mathbb{C}$ is defined to be a motive on each of the connected components of S.

HOPE 3.9. *For any representation (V,ξ) of G, there exists a motive $\mathcal{M}$ on $\mathrm{Sh}(G,X)$ such that*

$$\mathcal{H}_B(\mathcal{M}) = V(\xi), \quad \mathcal{H}_\ell(\mathcal{M}) = V_\ell(\xi) \text{ each } \ell, \quad \mathcal{H}_{\mathrm{dR}}(\mathcal{M}) = \mathcal{V}(\xi).$$

Take ξ to be faithful, and let $\mathcal{M}$ be the family of motives given by (3.9). There will be a family $\mathfrak{t} = (t_\alpha)$ of tensors for V such that G is the subgroup of $GL(V)\times\mathbb{G}_m$ fixing the t_α. For each α, t_α defines a global section s_α of $\mathcal{H}_B(\mathcal{M})$, and we let $\mathfrak{s} = (s_\alpha)$. By construction, there is an isomorphism $\eta : V_f(\xi) \to \mathcal{H}_f(\mathcal{M})$ sending t_α to s_α.

HOPE 3.10. *The triple $(\mathcal{M},\mathfrak{s},\eta)$ is universal: let S be a smooth $\mathbb{C}$-scheme with a continuous action of $G(\mathbb{A}_f)$, and let $(\mathcal{M}',\mathfrak{s}',\eta')$ be a triple over S such that $(\mathcal{M}',\mathfrak{s}',\eta')_s \in \mathcal{M}(G,X,\xi)$ for all closed points s of S; then there is a unique $G(\mathbb{A}_f)$-morphism $\Psi : S \to \mathrm{Sh}(G,X)$ such that $\Psi^*(\mathcal{M},\mathfrak{s},\eta) = (\mathcal{M}',\mathfrak{s}',\eta')$.*

Shimura varieties as moduli varieties for abelian varieties. We now drop all assumptions on motives. Let (G,X) be a pair satisfying (2.1), and assume that there is an inclusion $\xi : (G,X) \hookrightarrow (GSp,S^\pm)$, where GSp and $S^\pm$ are as in (2.4a). In this case, (3.4) is true; in fact, $(V, \xi \circ h_x)$ is the Hodge structure of an abelian variety A over $\mathbb{C}$, uniquely determined up isogeny. Thus $\mathcal{M}(G, X, \xi)$ consists of isogeny classes of triples $(A, \mathfrak{s}, \eta)$ satisfying (3.5), with A an abelian variety. (We say $(A, \mathfrak{s}, \eta)$ and $(A', \mathfrak{s}', \eta')$ are *isogenous* if there is an isogeny $\gamma : A \to A'$ sending s_α to s'_α, each α, and such that $\gamma \circ \eta = \eta'$.)

THEOREM 3.11. *(a) The map $\Phi_\xi : \mathcal{M}(G, X, \xi) \to \mathrm{Sh}(G, X)$ realizes $\mathrm{Sh}(G, X)$ as the coarse moduli scheme for the set $\mathcal{M}(G, X, \xi)$ of isogeny classes of triples $(A, \mathfrak{s}, \eta)$.*

(b) When (G, X) satisfies (2.1.2), $Sh(G, X)$ is a fine moduli scheme; in particular, it carries a universal family $(\mathcal{A}, \mathfrak{s}, \eta)$.*

PROOF: This follows from the main theorem of Mumford (1965).

A Shimura variety $\mathrm{Sh}(G, X)$ is said to be of *Hodge type* when there is an embedding $(G, X) \hookrightarrow (GSp(V, \psi), S^\pm)$. As we have just seen, every such Shimura variety is a (coarse) moduli scheme for abelian varieties with Hodge-cycle and level structure. When each of the Hodge cycles defining the moduli problem is an endomorphism or a polarization then the Shimura variety is said to be of *PEL-type*.

Notes. This section makes more explicit the philosophy underlying Deligne (1979).

4. Conjugates of Shimura varieties.

Let τ be an automorphism of $\mathbb{C}$. We want to identify $\tau\mathrm{Sh}(G, X)$ with the Shimura variety defined by a possibly different pair (G', X'). Fix a faithful representation (V, ξ) of G, and assume (3.4), so that attached to each point s of $\mathrm{Sh}(G, X)$, there is a triple $(M, \mathfrak{s}, \eta)_s$ satisfying the conditions (3.5). The triple attached to $\tau s \in \tau\mathrm{Sh}(G, X) = \mathrm{Sh}(G', X')$ should be $\tau(M, \mathfrak{s}, \eta)_s$. As we noted in (3.7) it is possible to recover (G', X') from $\tau(M, \mathfrak{s}, \eta)_s$. This gives us a description of (G', X'), but only in terms of a conjectural theory of motives. A key observation in Langlands (1979) is that, when we take s to be a CM-point, M_s becomes a CM-motive, and so we can apply the theory of the Taniyama group to define (G', X').

Now drop all assumptions, and choose a special point $x \in X$. Then x defines a homomorphism $\rho_x^{\mathrm{ad}} : \mathfrak{S} \to G^{\mathrm{ad}}$ (see §2), and hence an action of $\mathfrak{S}$ on G. Write ${}^{\tau,x}G$ for the inner twist of G defined by ${}^{\tau}\mathfrak{S}$: ${}^{\tau,x}G = {}^{\tau}\mathfrak{S} \times^{\mathfrak{S}} G$. The point $\mathrm{sp}(\tau) \in {}^{\tau}\mathfrak{S}(\mathbb{A}_f)$ defines an isomorphism

$$G(\mathbb{A}_f) \to {}^{\tau,x}G(\mathbb{A}_f), \quad g \mapsto {}^{\tau,x}g =_{\mathrm{df}} sp(\tau) \cdot g.$$

Let $T \subset G$ be a maximal torus such that $T(\mathbb{R})$ fixes x. The action of $\mathfrak{S}$on T is trivial, and so $T = {}^{\tau,x}T \subset {}^{\tau,x}G$. Thus $\tau\mu_x$ can be regarded as a homomorphism

$$\mathbb{G}_m \to T = {}^{\tau,x}T \hookrightarrow {}^{\tau,x}G.$$

Since $\tau\mu_x$ commutes with its complex conjugate, it defines a homomorphism $h_{\tau x} : \mathbb{S} \to {}^{\tau,x}G$, and when we take ${}^{\tau,x}X$ to be the set of $G(\mathbb{R})$-conjugates of ${}^{\tau,x}h$, the pair $({}^{\tau,x}G, {}^{\tau,x}X)$ satisfies the axioms (2.1).

Remark 4.1. (a) If x is a CM-point and (V, ξ) is a faithful representation of G, then, as we observed in §2, $(V, h_x \circ \xi)$ is $H_B(M)$ for a well-defined CM-motive M over $\mathbb{C}$. Let $\mathfrak{t}$ be a family of Hodge tensors for V such that G is the subgroup of $GL(V) \times \mathbb{G}_m$ fixing the elements of $\mathfrak{t}$. Then ${}^{\tau,x}G$ is the subgroup of $GL(H_B(\tau M)) \times \mathbb{G}_m$ fixing τt for each $t \in \mathfrak{t}$. Moreover, $h_{{}^\tau x} = h_{\tau M}$, and $g \mapsto {}^{\tau,x}g$ is defined by $H_f(M) \xrightarrow{\tau} H_f(\tau M)$.

(b) The group ${}^{\tau,x}G$ is obtained from G by twisting at infinity. For example, if $G = GL_1(B)$ with B a quaternion algebra over a totally real field F, then ${}^{\tau,x}G = GL_1(B')$ where $B' \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \approx B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, all ℓ, and $B' \otimes_{F,\sigma} \mathbb{R} \approx B \otimes_{F,\tau\circ\sigma} \mathbb{R}$, all $\sigma : F \hookrightarrow \mathbb{R}$.

The next result is the main theorem of the chapter: it shows that the choice of a special point x determines a realization of $\tau\mathrm{Sh}(G, X)$ as the Shimura variety of $({}^{\tau,x}G, {}^{\tau,x}X)$; the following Theorem 4.4 then shows that the realization is essentially independent of the choice of x.

THEOREM 4.2. *For each $\tau \in Aut(\mathbb{C})$ and special point $x \in X$, there is a unique isomorphism*

$$\varphi_{\tau,x} : \tau\mathrm{Sh}(G, X) \to \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$$

such that

(a) $\tau[x, 1] \mapsto [{}^\tau x, 1]$, *and*

(b) $\varphi_{\tau,x} \circ \tau\mathcal{T}(g) = \mathcal{T}({}^{\tau,x}g) \circ \varphi_{\tau,x}$, *all* $g \in G(\mathbb{A}_f)$.

PROOF: The uniqueness is obvious from the fact that $[x, 1] \cdot G(\mathbb{A}_f)$ is dense in $\mathrm{Sh}(G, X)$. We discuss the proof of the existence in §9 below. (If we knew (3.10), $\varphi_{\tau,x}$ would be the map given by the family of motives $\tau\mathcal{M}$ over $\tau\mathrm{Sh}(G, X)$ and the universality of $\mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$.)

Let x and x' be CM points of X (supposed to exist). A calculation shows that $\rho_{x*}({}^{\tau}\mathfrak{S})$ and $\rho_{x'*}({}^{\tau}\mathfrak{S})$ have the same class in $H^1(\mathbb{Q}, G)$. The choice of an isomorphism $f : \rho_{x*}({}^{\tau}\mathfrak{S}) \to \rho_{x'*}({}^{\tau}\mathfrak{S})$ determines an isomorphism $f_1 : {}^{\tau,x}G \to {}^{\tau,x'}G$, and there is an $a \in {}^{\tau,x}G(\mathbb{A}_f)$ such that $f_1(a^{-1} \cdot {}^{\tau,x}g) = {}^{\tau,x'}g$. If f is replaced by $f \circ q$, $q \in {}^{\tau,x}G(\mathbb{Q})$, then f_1 is replaced by $f_1 \circ \operatorname{ad} q$ and a with aq. Therefore (see 2.6), there is a well-defined isomorphism

$$\varphi(\tau; x', x) : \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X) \to \mathrm{Sh}({}^{\tau,x'}G, {}^{\tau,x'}X).$$

PROPOSITION 4.3. *Let $\tau \in Aut(\mathbb{C})$. For each pair (G, X) defining a Shimura variety and special points x and x' of X there is an isomorphism*

$$\varphi(\tau; x', x) : \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X) \to \mathrm{Sh}({}^{\tau,x'}G, {}^{\tau,x'}X)$$

such that $\varphi(\tau; x', x) \circ \mathcal{T}({}^{\tau,x}g) = \mathcal{T}({}^{\tau,x'}g) \circ \varphi(\tau; x', x)$, all $g \in G(\mathbb{A}_f)$. These isomorphisms are uniquely determined by the following properties:

(a) when x and x' are CM-points, $\varphi(\tau; x', x)$ is as defined above;
(b) if $(G, X)^+ = (G', X')^+$ and x and $x' \in X^+$ $(= X'^+)$, then

$$\varphi(\tau; x', x)|\mathrm{Sh}(G, X)^0 = \varphi(\tau; x', x)|\mathrm{Sh}(G', X')^0.$$

PROOF: When the weight w_X is defined over $\mathbb{Q}$, every special point is CM and the map is as above. Next check that $\varphi(\tau; x', x)|\mathrm{Sh}(G, X)^0 = \varphi(\tau; x', x)|\mathrm{Sh}(G', X')^0$ when $(G^{\mathrm{der}}, X^+) = (G'^{\mathrm{der}}, X'^+)$, x and x' both lie in X^+, and w_X and $w_{X'}$ are defined over $\mathbb{Q}$. In the general case, after possibly replacing x' by gx' with $g \in G(\mathbb{Q})$, we can assume that x and x' lie in the same connected component X^+ of X. Now (2.8) provides us with a pair (G', X') such that $(G', X')^+ = (G, X)^+$ and $w_{X'}$ is defined over $\mathbb{Q}$. Take $\varphi(\tau; x', x)$ to be the unique equivariant map whose restriction to $\mathrm{Sh}(G, X)^0$ is $\varphi(\tau; x', x)|\mathrm{Sh}(G', X')^0$.

THEOREM 4.4. *For any pair of special points x and x', we have $\varphi(\tau; x', x) \circ \varphi_{\tau,x} = \varphi_{\tau,x'}$:*

$$\begin{tikzcd}
 & \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X) \arrow[dd, "\varphi(\tau; x', x)"] \\
\tau\mathrm{Sh}(G, X) \arrow[ur, "\varphi_{\tau,x}"] \arrow[dr, "\varphi_{\tau,x'}"'] & \\
 & \mathrm{Sh}({}^{\tau,x'}G, {}^{\tau,x'}X).
\end{tikzcd}$$

PROOF: We discuss the proof in §9.

Remark 4.5. Let I be an index set. To give a family of objects $(S_i)_{i\in I}$ and isomorphisms $\varphi_{ji} : S_i \to S_j$, one for each pair (i,j), such that $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ for all i, j, k, is essentially the same as to give a single object: the inverse limit of the family is an object S together with isomorphisms $\varphi_i : S \to S_i$ such that $\varphi_{ji} \circ \varphi_i = \varphi_j$. From this point of view, Theorems 4.2 and 4.4 realize $\tau\mathrm{Sh}(G,X)$ as the inverse limit of the Shimura varieties $\mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$, x running over the special points of X.

Remark 4.6. Let K be a compact open subgroup of $G(\mathbb{A}_f)$, and let K' be the image of K in ${}^{\tau,x}G(\mathbb{A}_f)$ under $g \mapsto {}^{\tau,x}g$. Then $\varphi_{\tau,x}$ induces an isomorphism

$$\tau\mathrm{Sh}_K(G,X) \to \mathrm{Sh}_{K'}({}^{\tau,x}G, {}^{\tau,x}X).$$

Let f be a rational function on $\mathrm{Sh}_K(G,X)$ that is defined at the special point $[x,1]$. Then (4.2) associates with f a function ${}^\tau f =_{\mathrm{df}} \tau \circ f \circ \tau^{-1} \circ \varphi_{\tau,x}^{-1}$ on $S_{K'}({}^{\tau,x}G, {}^{\tau,x}X)$ such that

(i) ${}^\tau f([{}^\tau x, 1]) = \tau(f([x,1]))$

(ii) $f \mapsto {}^\tau f$ commutes with the Hecke operators.

This leads to a reciprocity law, which can be made more explicit (see Milne and Shih 1981b, §5).

Notes. Theorems (4.2) and (4.4) were conjectured by Langlands (Langlands 1979), who was motivated by the problem of computing the zeta function of a Shimura variety. For Shimura varieties of abelian type (see §9 for a definition of this class), they were proved in Milne and Shih (1982b), where also the proof of the general case was reduced to a statement about connected Shimura varieties defined by simply-connected simple groups. This statement was proved in Milne (1983) using a theorem of Kazhdan (1982) (whose proof is completed in Clozel (1986)) and theorems of Margulis (1977). See also Borovoi (1983/4) (completed in Borovoi (1987)) and the notes to §9 below.

5. Canonical models.

By a *model* of $\mathrm{Sh}(G,X)$ over a subfield E of $\mathbb{C}$, we mean a scheme S over E endowed with an action of $G(\mathbb{A}_f)$ (defined over E) and an equivariant isomorphism (over $\mathbb{C}$) $\psi : \mathrm{Sh}(G,X) \to S \otimes_E \mathbb{C}$. Note that ψ can also be regarded as morphism $\mathrm{Sh}(G,X) \to S$ over E inducing an isomorphism $\mathrm{Sh}(G,X) \to S \otimes_E \mathbb{C}$.

Let (T,x) be a special pair in (G,X). The field of definition of the cocharacter μ_x of T is the reflex field $E(T,x)$. As in (I.2.6), μ_x defines a $\mathbb{Q}$-rational homomorphism $N_x : T^E \to T$ for any field $E \supset E(T, x)$. The *reciprocity map*

$$r_E(T,x) : \mathrm{Gal}(E^{\mathrm{ab}}/E) \to T(\mathbb{A}_f)/T(\mathbb{Q})^-$$

is defined as follows: let $\tau \in \mathrm{Gal}(E^{\mathrm{ab}}/E)$, and let $s \in \mathbb{A}_E^\times$ be such that $\mathrm{rec}_E(s) = \tau^{-1}$; write $s = s_\infty \cdot s_f$ with $s_\infty \in E_\infty$ and $s_f \in \hat{E}$; then $r_E(T,x)(\tau) = N_x(s_f) \pmod{T(\mathbb{Q})^-}$.

Definition 5.1. A model $\mathrm{Sh}(G,X)_E$ of $\mathrm{Sh}(G,X)$ over $E = E(G,X)$ is said to be *canonical* if each special point $[x,a]$ is rational over $E(T,x)^{\mathrm{ab}}$) and $\mathrm{Gal}(E(T,x)^{\mathrm{ab}}/E(T,x))$ acts on $[x,a]$ according to the rule:

$$\tau[x,a] = [x, r(\tau)\cdot a], \quad \text{where } r = r_E(T,x).$$

PROPOSITION 5.2. *Consider a morphism $f : (G,X) \to (G',X')$. If $\mathrm{Sh}(G,X)$ and $\mathrm{Sh}(G',X')$ have canonical models, then the morphism $\mathrm{Sh}(f) : \mathrm{Sh}(G,X) \to \mathrm{Sh}(G',X')$ is defined over any field E containing the reflex fields of (G,X) and (G',X'), that is, there exists a (unique) morphism $\mathrm{Sh}(f)_E : \mathrm{Sh}(G,X)_E \to \mathrm{Sh}(G',X')_E$ making the following diagram commute:*

$$\begin{array}{ccc}
\mathrm{Sh}(f) : \mathrm{Sh}(G,X) & \longrightarrow & \mathrm{Sh}(G',X') \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi'} \\
\mathrm{Sh}(f)_E : \mathrm{Sh}(G,X)_E & \longrightarrow & \mathrm{Sh}(G',X')_E.
\end{array}$$

PROOF: See Deligne (1971c), 5.4.

COROLLARY 5.3. *The canonical model of $Sh(G,X)$ (if it exists) is uniquely determined up to a unique isomorphism.*

PROOF: This is an immediate consequence of the proposition.

Example 5.4. (a) Let T be a torus. Since $\mathrm{Sh}(T,x)$ is of dimension zero, it is completely described by its set of points (with the profinite topology), and so it has a unique model over $\mathbb{Q}^{\mathrm{al}}$. Giving a model of $\mathrm{Sh}(T,x)$ over $E = E(T,x)$ corresponds to giving an action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/E)$ on $\mathrm{Sh}(T,x)(\mathbb{Q}^{\mathrm{al}}) = T(\mathbb{A}_f)/T(\mathbb{Q})^-$. If the model is to be the canonical model, this action must be that given by $r(T,x)$.

(b) When (G,X) is of Hodge type, it follows from the theorem of Shimura and Taniyama (see I.5.6) that a solution to the moduli problem over $E(G,X)$ will be a canonical model.

THEOREM 5.5. *Let (G, X) be a pair satisfying (2.1), and write $E = E(G, X)$.*

(a) The Shimura variety $\mathrm{Sh}(G, X)$ *has a canonical model* $\mathrm{Sh}(G, X)_E$.

(b) For any $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, $\tau E(G, X) = E({}^{\tau,x}G, {}^{\tau,x}X)$, and $\tau\mathrm{Sh}(G, X)_E$ is the canonical model of $\mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$.

PROOF: This follows from (4.2) and (4.4). Suppose first that w_X is defined over $\mathbb{Q}$. A calculation shows that if τ fixes $E(G, X)$, then the class of $\rho_{x*}({}^{\tau}\mathfrak{S})$ in $H^1(\mathbb{Q}, G)$ is trivial. The choice of a point $p \in \rho_{x*}({}^{\tau}\mathfrak{S})$ determines an isomorphism $f_1 : G \to {}^{\tau,x}G$. Write $p = sp(\tau) \cdot \beta$. Then (2.6) give us a well-defined equivariant isomorphism

$$\varphi_x : \mathrm{Sh}(G, X) \to \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X).$$

A similar argument to that in the proof of (4.3) allows us to extend the definition of φ_x to any Shimura variety. For each τ, let

$$f_\tau = \varphi_x^{-1} \circ \varphi_{\tau,x} : \tau\mathrm{Sh}(G, X) \to \mathrm{Sh}(G, X).$$

Then $f_{\sigma\tau} = f_\sigma \circ \sigma(f_\tau)$, and so the f_τ define a descent datum for $\mathrm{Sh}(G, X)$ which gives us a model $\mathrm{Sh}(G, X)_E$ over $E(G, X)$. When applied to a pair (T, x), this procedure leads directly to the canonical model of $\mathrm{Sh}(T, x)$; thus $[x, a]$ is rational over $E(T, x)^{\mathrm{ab}}$, and the action of the Galois group on it is as required. Now (4.4) can be used to show that the model obtained is independent of the special point x, and so it fulfills the condition for every special point. This completes the proof of (a). The statement in (b) about the reflex fields is obvious from the definitions. Moreover, it is straightforward to check that

$$(\tau\mathrm{Sh}(G, X)_E) \otimes_{\tau E} \mathbb{C} = \tau\mathrm{Sh}(G, X) \xrightarrow{\varphi_{\tau,x}} \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$$

realizes $\tau\mathrm{Sh}(G, X)_E$ as the canonical model of $\mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$.

COROLLARY 5.6. *Let $E = E(G, X)$. Then*

$$\prod_{\tau \in \mathrm{Hom}(E, \mathbb{C})} \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$$

has a canonical model over $\mathbb{Q}$.

PROOF: In fact, the maps $\varphi_{\tau,x}$ define an isomorphism

$$(\mathrm{Res}_{E/\mathbb{Q}}\mathrm{Sh}(G, X)_E)_{\mathbb{C}} \to \prod \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X).$$

For any field L containing $E(G, X)$, $\mathrm{Sh}(G, X)_E$ gives rise to a model $\mathrm{Sh}(G, X)_L$ of $\mathrm{Sh}(G, X)$ over L. This model will be referred to as the *canonical model of* $\mathrm{Sh}(G, X)$ *over* L.

Notes. Canonical models (in the above sense) were introduced, and shown to be unique in Deligne (1971c). Again, the notion was suggested by a similar notion introduced by Shimura (see the next section). They were shown to exist for Shimura varieties of abelian type (see §9) in Deligne (1979). That (4.2) and (4.4) imply the existence of canonical models was already noted in Langlands (1979).

6. Canonical models in the sense of Shimura.
According to Shimura's original definition, the canonical model of a Shimura variety should be a projective system of *connected* varieties. We explain how such models can be constructed from the canonical models of the preceding section.

Let (G, X) be a pair satisfying (2.1), and choose a connected component X^+ of X. The canonical model (in the sense of Shimura) will be defined in terms of the pair (G, X^+) — note that this is not a pair satisfying (1.3) — G is a *reductive* group. Write $\mathrm{Sh}(G, X)^0$ for the connected component of $\mathrm{Sh}(G, X)$ containing the image of X^+, and let E be the reflex field of (G, X). Since $\mathrm{Sh}(G, X)$ has a canonical model over E, there is a homomorphism $\ell : \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/E) \to \pi_0(\mathrm{Sh}(G, X))$ giving the action of the Galois group on the set of connected components of $\mathrm{Sh}(G, X)$ (see Deligne (1979), 2.6.2.1, for an explicit description of ℓ). According to (2.7), there is an exact sequence

$$1 \longrightarrow G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge} \longrightarrow \mathcal{G}(G) \longrightarrow \pi_0(\mathrm{Sh}(G, X)) \longrightarrow 1.$$

On pulling back by ℓ, we obtain a sequence

$$1 \longrightarrow G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge} \longrightarrow \mathcal{E}(G, X) \xrightarrow{\sigma} \mathrm{Gal}(\mathfrak{k}/E) \longrightarrow 1$$

with $\mathrm{Gal}(\mathfrak{k}/E)$ the image of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/E)$ in $\pi_0(\mathrm{Sh}(G, X))$ and $\mathcal{E}(G, X)$ the subgroup of $\mathcal{G}(G)$ of elements mapping to $\mathrm{Gal}(\mathfrak{k}/E)$). From $\mathrm{Sh}(G, X)_E$ we obtain a canonical model $\mathrm{Sh}(G, X)^0_{\mathfrak{k}}$ of $\mathrm{Sh}(G, X)^0$ over $\mathfrak{k}$.

Let $\mathfrak{z}$ be the set of compact open subgroups of $\mathcal{E}(G, X)$. For any S in $\mathfrak{z}$, set

$\Gamma_S = S \cap G^{\mathrm{ad}}(\mathbb{Q})^+$;

$k_S =$ the subfield of $\mathfrak{k}$ fixed by $\sigma(S)$;

$V_S = S \backslash \mathrm{Sh}(G, X)^0$; it is defined over k_S, and there is an isomorphism $\varphi_S : \Gamma_S \backslash X^+ \to (V_S)_{\mathbb{C}}$. Let $\alpha \in \mathcal{E}(G, X)$; if $\alpha S \alpha^{-1} \subset T$, then the action of α on $\mathrm{Sh}(G, X)^0$ induces a map $J_{TS}(\alpha) : V_S \to \sigma(\alpha)^{-1} V_T$.

THEOREM 6.1. *(a) For each $S \in \mathfrak{Z}$, (V_S, φ_S) is a model of $\Gamma_S\backslash X^+$ over k_S.*

(b) Let $\alpha \in \mathcal{E}(G,X)$; for any S, $T \in \mathfrak{k}$ such that $\alpha S\alpha^{-1} \subset T$, $J_{TS}(\alpha) : \Gamma_S\backslash X^+ \to \sigma(\alpha)^{-1}\Gamma_T\backslash X^+$ is defined over k_S. Moreover

$J_{SS}(\alpha)$ is the identity map if $\alpha \in S$;

$(\sigma(\alpha)^{-1}J_{TS}(\beta)) \circ J_{SR}(\alpha) = J_{TR}(\beta\alpha)$;

$J_{TS}(\alpha) \circ \varphi_S = \varphi_T \circ \alpha$ for all $\alpha \in G(\mathbb{Q})_+$ such that $\alpha S\alpha^{-1} \subset T$.

(c) Let $x \in X^+$ be special; for each $S \in \mathfrak{Z}$, $\varphi_S(z)$ is rational over $E(x)^{\mathrm{ab}}$, and for every $\nu \in \hat{E}(x)^\times$,

$$\mathrm{rec}_E(\nu)(\varphi_S(x)) = J_{ST}(N_x(\nu))\varphi_T(x), \quad T = N_x(\nu)^{-1} \cdot S \cdot N_x(\nu)$$

where $N_x : T^{E(x)} \to G$ is defined by μ_x.

PROOF: This can be deduced from (5.5a), using results about the automorphism groups of $\mathrm{Sh}(G,X)$ and its function field. See Milne and Shih (1981b).

Notes. Theorem 6.1 says that canonical models exist in the sense of Shimura (1971a). It was proved in various cases in Shimura (1970), Miyake (1971), and Shih (1978). It was shown to follow from Theorem 5.5 in Milne and Shih (1981b) (the restriction to classical groups in that paper is unnecessary).

7. The action of complex conjugation on a Shimura variety with a real canonical model.

Let $\mathrm{Sh}(G,X)$ be a Shimura variety whose reflex $E(G,X)$ is real. Then $\mathrm{Sh}(G,X)$ has a canonical model $\mathrm{Sh}(G,X)_{\mathbb{R}}$ over $\mathbb{R}$, and so complex conjugation defines an involution of $\mathrm{Sh}(G,X)$. In order to be able to compute the factor of the zeta function of $\mathrm{Sh}(G,X)$ corresponding to the (given) infinite prime of $E(G,X)$, it is necessary to have an explicit description of this involution.

LEMMA 7.1. *Let x be a special point of X. There is a unique $G(\mathbb{R})$-equivariant antiholomorphic map $X \to X$ such that $\eta(x) = {}^\iota x$, where ${}^\iota x$ is the point in X such that $\mu_{{}^\iota x} = \iota\mu_x$.*

PROOF: The uniqueness is obvious. Let T be a maximal torus in G such that $T(\mathbb{R})$ fixes x, and let N be the normalizer of T in G. There is an $n \in N(\mathbb{R})$ such that $n \cdot x = {}^\iota x$ (Milne and Shih 1982b, 4.3), and we can define η to be $g \cdot x \mapsto gn \cdot x$.

THEOREM 7.2. *Let* $\mathrm{Sh}(G,X)$ *be a Shimura variety whose reflex field is real. The involution of* $\mathrm{Sh}(G,X)$ *defined by complex conjugation is* $[x,g]\mapsto[\eta(x),g]$.

PROOF: Since both maps are continuous and equivariant, it suffices to show that they agree at the single point $[x,1]$. The action of ι on $\mathrm{Sh}(G,X)$ (relative to $\mathrm{Sh}(G,X)_E$ is

$$\mathrm{Sh}(G,X)\xrightarrow{\iota}\iota\mathrm{Sh}(G,X)\xrightarrow{\varphi_{\iota,x}}\mathrm{Sh}({}^{\iota,x}G,{}^{\iota,x}X)\xrightarrow{\varphi_x^{-1}}\mathrm{Sh}(G,X).$$

From §5 and §6, we see that $[x,1]\mapsto\iota[x,1]\mapsto[{}^{\iota}x,1]\mapsto[\eta(x),1]$ under these maps.

Notes. Theorem 7.2 was conjectured in Langlands (1979). An equivalent statement for connected Shimura varieties defined by groups G of type C was proved in Shih (1976), and this result was extended to all Shimura varieties of abelian type in Milne and Shih (1981a). That Theorem 7.2 follows from Theorems 4.2 and 4.4 was noted in Langlands (1979).

8. The minimal compactification.

Let $\mathrm{Sh}(G,X)^-$ be the minimal compactification of $\mathrm{Sh}(G,X)$. Because $\mathrm{Sh}(G,X)^-$ can be constructed out of $\mathrm{Sh}(G,X)$ by a canonical algebraic method (see §2), all the maps $\varphi_{\tau,x}$, $\varphi(\tau;x',x)$ and φ_x have unique extensions to $\mathrm{Sh}(G,X)^-$. In particular, we see that all the theorems in this chapter remain valid when the Shimura varieties are replaced by their minimal compactifications. (We shall discuss the boundary components of $\mathrm{Sh}(G,X)^-$in more detail in Chapter V.)

9. The strategy for proving the main theorems.

The proofs of Theorems 4.2 and 4.4 are too long to describe in detail. Instead I outline the strategy for proving them, and other theorems, on Shimura varieties. Recall that in §3 we defined the notion of a Shimura variety of Hodge type and noted that the choice of a faithful representation of G realizes such a variety as a moduli variety (over $\mathbb{C}$) for abelian varieties with Hodge cycle and level structure.

The class of connected Shimura varieties of *abelian type* is the smallest containing:

(a) the connected component of every Shimura variety of Hodge type;

(b) a product of connected Shimura varieties if it contains the factors;

(c) $\mathrm{Sh}(G, X^+)$ if it contains $\mathrm{Sh}(G', X^+)$ with G' a finite covering group of G.

Deligne (1979) gives a classification of connected Shimura varieties of abelian type based on Satake's classification of symplectic embeddings (Satake 1965). A (nonconnected) Shimura variety is of *abelian type* if a connected component of it is of abelian type. Note that a Shimura variety of abelian type will *not* in general be a moduli variety for abelian varieties, contrary to some assertions in the literature.

Let $P(G, X)$ be a statement about the Shimura variety $\mathrm{Sh}(G, X)$. The first step in proving P for all Shimura varieties is to prove it for those of Hodge type by identifying the Shimura variety with a moduli variety for abelian varieties. The second step is to find a statement $P^+(G, X^+)$ for connected Shimura varieties, and to prove that

$$P(G, X) \text{ is true } \Leftrightarrow P^+(G^{\mathrm{der}}, X^+) \text{ is true.}$$

As a consequence, one finds that if $P(G', X')$ is true and $(G^{\mathrm{der}}, X^+) = (G'^{\mathrm{der}}, X'^+)$, then $P(G, X)$ is true. The third step (usually easy) is to prove:

$P^+(G_i, X_i^+)$ true for all $i \Rightarrow P^+(\prod G_i, \prod X_i^+)$ true;

$P^+(G', X^+)$ true for G' a finite covering of $G \Rightarrow P^+(G, X^+)$ is true.

This then implies that P^+ is true for all connected Shimura varieties of abelian type, and hence (by the previous step) that P is true for all Shimura varieties of abelian type. Moreover, it shows that in order to prove P for all Shimura varieties, it suffices to prove $P^+(G, X^+)$ in the case that G is a simply connected $\mathbb{Q}$-simple group. Then G is of the form $G = \mathrm{Res}_{F/\mathbb{Q}} G'$ for some absolutely simple group G' over a totally real field F. For a totally real field F' containing F, set $G_* = \mathrm{Res}_{F'/\mathbb{Q}} G$ and define X_*^+ so that $(G, X^+) \subset (G_*, X_*^+)$. When F' is chosen sufficiently large, there will be many embeddings $(G_\alpha, X_\alpha^+) \hookrightarrow (G, X^+)$ with G_α a group of type A_1 (thus G_α is an algebraic group associated with a quaternion algebra, possibly split, over a totally real field). We have

$$\mathrm{Sh}(G, X^+) \hookrightarrow \mathrm{Sh}(G_*, X_*^+) \hookleftarrow \mathrm{Sh}(G_\alpha, X_\alpha^+).$$

The final step is to exploit these inclusions, and the fact that the statement $P^+(G_\alpha, X_\alpha^+)$ is known (the associated Shimura variety is of abelian type), to prove $P^+(G, X^+)$.

One final note: several authors have criticized the above approach for its dependence on abelian varieties and their moduli. In defence

I point out that, in the case that the weight is defined over $\mathbb{Q}$, all of the results in this and the next chapter would be an immediate consquence of the existence of a sufficiently strong theory of motives and their moduli; moreover, this is the only heuristic argument I know for them. Also, the approach does not use the classification of semisimple algebraic groups (at present, the only place where this is used is in Kazhdan (1982), but the author has shown that it is unnecessary there). Finally, this is the *only* approach that gives strong results.

Notes. For the existence of canonical models, the first three steps were carried out in Deligne (1979). For Langlands's conjecture (theorems 4.2 and 4.4) they were carried out in Milne and Shih (1982b). The embedding of $\mathrm{Sh}(G,X)$ into $\mathrm{Sh}(G_*,X_*)$ was used in Piatetski-Shapiro (1971) in the case the group G is of type A_n to obtain a pair (G_*,X_*) for which $G_*(\mathbb{Q}_\ell)$ has no compact factors. Borovoi suggested (in 1981) using the embeddings $(G_\alpha,X_\alpha)\hookrightarrow(G_*,X_*)$ to prove the existence of canonical models for Shimura varieties not of abelian type. (Obtaining canonical models using embeddings of Shimura varieties of type A_1 was also an unstated object of Garrett (1982, 1984).)

10. Appendix: Schemes with a continuous action of a locally profinite group.

A *locally profinite group* is a locally compact totally disconnected group. In such a group $\mathcal{G}$, the compact open subgroups K form a fundamental system of neighbourhoods of the identity element, and $\cap K=1$.

LEMMA 10.1. *Let $\mathcal{G}$ be a locally profinite group, and let E be a separated topological space with a continuous action $E\times\mathcal{G}\to E$ of $\mathcal{G}$. For each compact open subgroup K in $\mathcal{G}$, set $E_K=E/K$. Then (E_K) is a projective system, and $E=\varprojlim E/K$.*

PROOF: Apply Bourbaki (1960), III.7.2, Cor 1 to the groups K acting on E, and observe that $\varprojlim K=\cap K=1$.

To give E together with the action of $\mathcal{G}$ is the same as to give the family (E_K) together with the maps

$$x\mapsto xg: E_K\to E_L,\quad L\supset g^{-1}Kg.$$

These remarks motivate the following definitions.

For the remainder of this section, "scheme" will mean "quasi-projective scheme over a field k", or a projective limit of such schemes.

Let $\mathcal{G}$ be a locally profinite group, and consider a family (S_K) of schemes, indexed by the open compact subgroups K of $\mathcal{G}$. Suppose that for each $g \in \mathcal{G}$ and each K and L with $L \supset g^{-1}Kg$, there is given a morphism

$$\rho_{L,K}(g) : S_K \to S_L$$

satisfying the conditions:

(i) $\rho_{K,K}(k) = id$ if $k \in K$;

(ii) $\rho_{M,L}(g) \circ \rho_{L,K}(h) = \rho_{M,L}(gh)$;

(iii) whenever K is normal in L, so that $\rho_{K,K}$ defines an action of the finite group L/K on S_K, S_L is isomorphic to the quotient of S_K by the finite group L/K.

We then call the family $(S_K, \rho_{L,K})$ a *scheme with a continuous right action* of $\mathcal{G}$.

For each $K \subset L$, there is a map $\rho_{L,K}(1) : S_K \to S_L$. In this way we get a projective system of schemes whose limit S has a right action by $\mathcal{G}$ such that $S_K = S/K$ for all compact open subgroups K of $\mathcal{G}$. We shall also refer to S as a *scheme with a continuous right action* of $\mathcal{G}$.

Example 10.2. Suppose $\mathcal{G}$ is compact and S is smooth. If $\mathcal{G}$ acts continuously on S in such a way that the isotropy group of each geometric point of S is trivial, then $S \to S/\mathcal{G}$ is a Galois covering with Galois group $\mathcal{G}$. Conversely, if $S \to S_0$ is a Galois covering with Galois group $\mathcal{G}$, then $\mathcal{G}$ acts on S in such a way that the isotropy group of each geometric point is trivial.

For example, let S_0 be a connected scheme, and fix a geometric point $s \to S_0$. Take S to be the projective system of commutative triangles

$$\begin{tikzcd} s \arrow[r] \arrow[dr] & S' \arrow[d, "\pi"] \\ & S_0 \end{tikzcd}$$

with π finite and étale. Then S is a Galois covering of S_0 with Galois group the étale fundamental group $\pi_1^{\text{et}}(S_0, s)$.

Let S be a scheme over k with a continuous action of $\mathcal{G}$. For a scheme Y over k, we set $\text{Hom}(Y, S) = \varinjlim \text{Hom}(Y, S_K)$. To give a scheme Y over S is the same as to give a scheme Y_K over S_K, each K sufficiently small, such that $Y_K = Y_L \times_{S_L} S_K$, for $K \subset L$.

Fix a locally profinite group $\mathcal{G}$ and a profinite set π with a continuous right action of $\mathcal{G}$. Assume that the action is transitive, and that the orbits of a compact open subgroup are open: for any $e \in \pi$, the bijection $\mathcal{G}/\mathcal{G}_e \to \pi$ ($\mathcal{G}_e$ = isotropy group at e) is a homeomorphism. Consider systems consisting of a scheme S with a continuous right action of $\mathcal{G}$ together with a continuous equivariant map $S \to \pi$. For $e \in \pi$, the fibre S_e over e is endowed with a continuous action of $\mathcal{G}_e$.

PROPOSITION 10.3. *The functor $S \mapsto S_e$ is an equivalence from the category of schemes S, endowed with a continuous action of $\mathcal{G}$ and a continuous equivariant map $S \to \pi$, to the category of schemes S_e endowed with a continuous action of $\mathcal{G}_e$.*

PROOF: See Deligne 1979, 2.7.3.

In particular, there is a reverse functor, $S_e \mapsto S$. The scheme S will be said to have been obtained from S_e by *induction* from $\mathcal{G}_e$ to $\mathcal{G}$.

III. AUTOMORPHIC VECTOR BUNDLES

Just as automorphic functions are sections of the sheaf of germs of functions on a Shimura variety, holomorphic automorphic forms are sections of certain vector bundles, called automorphic vector bundles, on a Shimura variety. The main theorems for automorphic vector bundles parallel those for Shimura varieties: every automorphic vector bundle $\mathcal{V}(\mathcal{J})$ has a canonical model $\mathcal{V}(\mathcal{J})_E$ over its reflex field E, and for each $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, $\tau\mathcal{V}(\mathcal{J})_E$ is the canonical model over τE of an explicitly determined automorphic vector bundle $\mathcal{V}({}^\tau\mathcal{J})$. In particular, this allows us to define, in complete generality, the notion of a holomorphic automorphic form being rational over a number field.

Throughout this chapter (G,X) is a pair satisfying (II.2.1). We write $Z_s(G)$ for the maximal subtorus of $Z(G)$ that is split over $\mathbb{R}$ but which has no subtorus split over $\mathbb{Q}$; thus $Z_s(G)$ is the largest subtorus of $Z(G)$ such that

$$\begin{cases} X_*(Z_s)^{\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})} = 0 \\ \iota \text{ acts as } +1 \text{ on } X_*(Z_s). \end{cases}$$

We write G^c for $G/Z_s(G)$. Note that (G,X) satisfies (II.2.1.2*) if and only if $G = G^c$.

1. The compact dual symmetric Hermitian space $\check{X}$. For each $x \in X$, μ_x defines a decreasing filtration $\mathrm{Filt}(\mu_x)$ of $\mathbf{Rep}_{\mathbb{C}}(G)$ (see I.1), and we define $\check{X}$ to be the $G(\mathbb{C})$-conjugacy class of filtrations of $\mathbf{Rep}_{\mathbb{C}}(G)$ containing $\mathrm{Filt}(\mu_x)$. If (V, ξ) is a faithful representation of $G_{\mathbb{C}}$, then $\check{X}$ can be identified with a $G(\mathbb{C})$-conjugacy class of filtrations of V.

Fix a point o of X, and let P_o be the subgroup of $G_{\mathbb{C}}$ fixing $\mathrm{Filt}(\mu_o)$. Then P_o is a parabolic subgroup of $G_{\mathbb{C}}$ (see I.1.7) and there is a bijection

$$G(\mathbb{C})/P_o(\mathbb{C}) \to \check{X},$$

which endows $\check{X}$ with the structure of a smooth projective variety over $\mathbb{C}$. We call $\check{X}$ the *compact dual symmetric Hermitian space* of X. For any connected component X^+ of X, $\check{X}$ is the dual of X^+ in the sense of Helgason (1978), V.2.

Interpretation of $\check{X}$ as a classifying space. Let $\mathcal{V}$ be a vector bundle on a connected complex variety S. The *type* of a filtration

$$\mathcal{V} \supset \mathcal{S}_1 \supset \cdots \supset \mathcal{S}_n = 0$$

is the sequence of numbers, $d_i = \operatorname{rank} \mathcal{S}_i$. Fix a vector space V over $\mathbb{C}$ and a filtration $F_0^{\cdot}$ of V of type $\mathbf{d} = (d_1, \ldots, d_n)$. Then the functor of complex varieties

$$\mathcal{F}(S) = \{\text{filtrations of } \mathcal{V}_S =_{\mathrm{df}} S \times V \text{ of type } \mathbf{d}\}$$

is represented by the Grassman variety $GL(V)/Q_0$, where Q_0 is the subgroup of $GL(V)$ stabilizing $F_0^{\cdot}$. When V is defined over $\mathbb{Q}$, so also is the Grassman variety.

Fix a family of tensors $\mathfrak{t} = (t_\alpha)_{\alpha \in I}$ for V, and let G be the subgroup of $GL(V)$ fixing the t_α. Then each t_α defines a global tensor of $\mathcal{V}_S$, and the functor

$$\mathcal{F}_0(S) = \{\text{filtrations } F^{\cdot} \text{ of } \mathcal{V}_S \text{ s.t. } (\mathcal{V}_s, F_s^{\cdot}, \mathfrak{t}) \approx (V, F_0^{\cdot}, \mathfrak{t}) \text{ all } s \in S\}$$

is represented by the subvariety G/P_0 of the Grassman variety, where P_0 is now the stabilizer of $F_0^{\cdot}$ in G.

We apply these remarks to (G, X). Choose a faithful representation $\xi : G \to GL(V)$ of G, and let $\mathfrak{t} = (t_\alpha)$ be a family of tensors of V such that G is the subgroup of $GL(V)$ fixing the t_α. Choose a point $o \in X$, and let $F_o^{\cdot}$ be the corresponding Hodge filtration of $V(\mathbb{C})$. Then $\check{X}$ represents the functor $\mathcal{F}_0$ described above: the $F_x^{\cdot}$ for $x \in \check{X}$ define a filtration of the vector bundle $\mathcal{V} =_{\text{df}} \check{X} \times V(\mathbb{C})$ and the triple $(\mathcal{V}, F^{\cdot}, \mathfrak{t})$ is universal.

In particular, we see that $\check{X}$ is realized as a subvariety of a Grassman variety $GL(V(\mathbb{C}))/Q_o$. As in (II.2), we let M_X be the $G(\mathbb{C})$-conjugacy class of homomorphisms $\mathbb{G}_m \to G_{\mathbb{C}}$ containing μ_x for $x \in X$, and we let $E(G, X)$ be the field of definition of M_X. The map

$$\mu_x \mapsto \mathrm{Filt}(\mu_x) : M_X \to \check{X}$$

is surjective, from which it is clear that $\check{X}$, regarded as a subvariety of $GL(V)/Q_0$, is stable under the action of any automorphism τ fixing $E(G, X)$. Therefore $\check{X}$ is defined over $E(G, X)$.

The Borel embedding.

PROPOSITION 1.1. *The map*

$$\beta : X \to \check{X}, \quad x \mapsto \mathit{Filt}(\mu_x)$$

embeds X as an open complex submanifold of $\check{X}$. For $o \in X$, let K_o be the isotropy group at o in $G(\mathbb{R})$, and let P_o be the isotropy group at $o \in \check{X}$ in $G(\mathbb{C})$; then $K_o = P_o \cap G(\mathbb{R})$, and the inclusion of K_o into P_o identifies $(K_o)_{\mathbb{C}}$ with a Levi subgroup of P_o; we have

$$\begin{array}{ccc} G(\mathbb{R})/K_o & \hookrightarrow & G(\mathbb{C})/P_o(\mathbb{C}) \\ \downarrow \approx & & \downarrow \approx \\ X & \hookrightarrow & \check{X}. \end{array}$$

PROOF: The fact that β is holomorphic is a restatement of (II.3.2a). For the rest, we merely note that the injectivity of $X \to \check{X}$ follows from the fact that the Hodge filtration determines the Hodge decomposition (I.2). (See Helgason 1978, VIII.7 for the details.)

The map β is the *Borel embedding* of X into $\check{X}$.

Example 1.2. Let (V, ψ) be a symplectic space, and let $(G, S^{\pm})$ be as in (II.2.4a). Thus $G = GSp(V, \psi)$ and $S^{\pm}$ is the space of Hodge structures on V of type $\{(-1,0),(0,-1)\}$ for which $\pm(2\pi i)\psi$ is a polarization. In this case, $\check{X}$ can be identified with the set of maximal isotropic subspaces of $V(\mathbb{C})$ and β with the map $x \mapsto F_x^0 V$.

Conjugates of $\check{X}$. As $\check{X}$ is an algebraic variety, $\tau\check{X}$ is defined for any $\tau \in \mathrm{Aut}(\mathbb{C})$. Recall from (I.7) that the period torsor $\mathfrak{P}$ is a torsor for $\mathfrak{T}$ having a canonical point $p \in \mathfrak{P}(\mathbb{C})$. Define $z_\infty(\tau) \in \mathfrak{T}(\mathbb{C})$ by:

$$\tau p = p \cdot z_\infty(\tau).$$

Then $z_\infty(\tau) \in {}^\tau\mathfrak{S}(\mathbb{C})$, and so it defines an isomorphism

$$g \mapsto {}^{\tau,x}g =_{\mathrm{df}} [z_\infty(\tau) \cdot g] : G(\mathbb{C}) \to {}^{\tau,x}G(\mathbb{C}).$$

PROPOSITION 1.3. *(a) Let x be a special point of X, and let ${}^{\tau,x}\check{X}$ be the dual Hermitian symmetric space associated with $({}^{\tau,x}G, {}^{\tau,x}X)$. There is a unique isomorphism*

$$\check{\varphi}_{\tau,x} : \tau\check{X} \to {}^{\tau,x}\check{X}$$

such that

(i) the point τx is mapped to ${}^\tau x$, and

(ii) $\check{\varphi}_{\tau,x} \circ (\tau g) = ({}^{\tau,x}g) \circ \check{\varphi}_{\tau,x}$, for all g in $G(\mathbb{C})$.

(b) Let x' be a second special point; then the isomorphism

$${}^{\tau,x}g \mapsto {}^{\tau,x'}g : {}^{\tau,x}G(\mathbb{C}) \to {}^{\tau,x'}G(\mathbb{C})$$

induces an isomorphism $\check{\varphi}(\tau; x', x) : {}^{\tau,x}\check{X} \to {}^{\tau,x'}\check{X}$ such that

$$\check{\varphi}(\tau; x', x) \circ \check{\varphi}_{\tau,x} = \check{\varphi}_{\tau,x'}.$$

PROOF: Straightforward.

Remark 1.4. Let x be a CM-point of X, and let (V, ξ) be a faithful representation of G. Then $(V, \xi \circ \rho_x)$ defines a CM-motive M over $\mathbb{Q}^{\mathrm{al}}$ with $V = H_B(M)$. There are Hodge cycles t_α on M such that G is the subgroup of $GL(V) \times \mathbb{G}_m$ fixing the t_α, and we noted in (II.4.1) that ${}^{\tau,x}G$ is the subgroup of $GL(H_B(\tau M)) \times \mathbb{G}_m$ fixing the tensors τt_α. The comparison isomorphisms between Betti and de Rham cohomology allow us to interprete $G_{\mathbb{C}}$ and ${}^{\tau,x}G_{\mathbb{C}}$ as subgroups of $GL(H_{\mathrm{dR}}(M_{\mathbb{C}})) \times \mathbb{G}_m$ and $GL(H_{\mathrm{dR}}(\tau M_{\mathbb{C}})) \times \mathbb{G}_m$ respectively. If we regard τ as an embedding of $\mathbb{Q}^{\mathrm{al}}$ into $\mathbb{C}$, then the map $G(\mathbb{C}) \to {}^{\tau,x}G(\mathbb{C})$ is induced by the isomorphism

$$H_{\mathrm{dR}}(M) \otimes_{\mathbb{Q}^{\mathrm{al}},\tau} \mathbb{C} \to H_{\mathrm{dR}}(\tau M).$$

2. Automorphic vector bundles.

Let S be an algebraic variety over a field k with an action $G \times S \to S$ of an algebraic group. By a *G-vector bundle* on S we mean a vector bundle $(\mathcal{V}, p)$ on S together with an action of G on $\mathcal{V}$ (as an algebraic variety) such that

(a) $p(g \cdot v) = g \cdot p(v)$ for all $g \in G$, $v \in \mathcal{V}$;
(b) the maps $g : \mathcal{V}_s \to \mathcal{V}_{gs}$ are linear for all $s \in S$.

We shall be interested in $G_{\mathbb{C}}$ vector bundles $\mathcal{J}$ on $\check{X}$. As we saw in (1.1), the map $\beta : X \hookrightarrow \check{X}$ embeds X as an open submanifold of $\check{X}$, and the action of $G(\mathbb{C})$ on $\check{X}$ extends that of $G(\mathbb{R})$ on X. Therefore such a vector bundle $\mathcal{J}$ restricts to a $G(\mathbb{R})$-vector bundle $\beta^{-1}(\mathcal{J})$ on X. If the action of $G_{\mathbb{C}}$ on $\mathcal{J}$ factors through $G^c_{\mathbb{C}}$, and K is sufficiently small, then, as in the proof of (II.3.3), we can pass to the quotient and obtain a vector bundle

$$\mathcal{V}_K(\mathcal{J}) = G(\mathbb{Q})\backslash\beta^{-1}(\mathcal{J}) \times G(\mathbb{A}_f)/K$$

on $\mathrm{Sh}(G, X)$. (In §8 we discuss what happens when we no longer require that the action factors through G^c.) For each $g \in G(\mathbb{A}_f)$ and pair of open compact subgroups K and L of $G(\mathbb{A}_f)$ such that $L \supset g^{-1}Kg$, there is a morphism

$$\rho_{L,K}(g) : \mathcal{V}_K(\mathcal{J}) \to \mathcal{V}_L(\mathcal{J}), \quad [x, a] \mapsto [x, ag].$$

PROPOSITION 2.1. *(a) The vector bundles $\mathcal{V}_K(\mathcal{J})$, and the maps $\rho_{L,K}(g) : \mathcal{V}_K(\mathcal{J}) \to \mathcal{V}_L(\mathcal{J})$, are algebraic.*

(b) If X^+ has no factors isomorphic to the unit disk, then every analytic section of $\mathcal{V}_K(\mathcal{J})$ is algebraic, and the space of such sections is finite-dimensional over $\mathbb{C}$.

PROOF: (a) When the boundary of $\mathrm{Sh}(G, X)$ in its minimal compactification has codimension ≥ 3, the proposition is a consequence of the following general lemma. We omit the proof in the remaining case (but see (3.6) below).

(b) The hypothesis implies that the codimension of the boundary is ≥ 2, and so the next lemma applies.

LEMMA 2.2. *Let S be a nonsingular algebraic variety over $\mathbb{C}$, embedded as an open subvariety of a complete algebraic variety $\bar{S}$. If $\bar{S} - S$ has codimension ≥ 2, then the functor $\mathcal{V} \mapsto \mathcal{V}^{\mathrm{an}}$ taking an algebraic vector bundle on S to its associated analytic vector bundle* *is fully faithful; moreover* $\Gamma(S, \mathcal{V}) = \Gamma(S, \mathcal{V}^{\mathrm{an}})$ *and these spaces are finite-dimensional. If* $\bar{S} - S$ *has codimension* ≥ 3, *then* $\mathcal{V} \mapsto \mathcal{V}^{\mathrm{an}}$ *is an equivalence of categories.*

PROOF: This follows from theorems of Serre, Grothendieck, Siu, and Trautmann; see Hartshorne (1970), p222.

The family $\mathcal{V}(\mathcal{J}) = (\mathcal{V}_K(\mathcal{J}))_K$ is a scheme with a right action of $G(\mathbb{A}_f)$, in the sense of (II.10). A vector bundle of the form $\mathcal{V}_K(\mathcal{J})$, $\mathcal{J}$ a $G_{\mathbb{C}}$-vector bundle on $\check{X}$, will be called an *automorphic vector bundle,* and a section f of $\mathcal{V}(\mathcal{J})_K$ over $\mathrm{Sh}_K(G, X)$ will be called a *(holomorphic) automorphic form* of type $\mathcal{J}$ and level K. (When the boundary of $\mathrm{Sh}(G, X)$ in its minimal compactification has codimension one we must also require that f be "holomorphic at infinity".)

Remark 2.3. (a) Fix a point $o \in \check{X}$, and let P_o be the (parabolic) subgroup of $G_{\mathbb{C}}$ fixing o. For any $G_{\mathbb{C}}$-vector bundle $\mathcal{J}$ on $\check{X}$, P_o acts on the fibre $\mathcal{J}_o$, and the map $\mathcal{J} \mapsto \mathcal{J}_o$ defines an equivalence from the category of $G_{\mathbb{C}}$-vector bundles on $\check{X}$ to $\mathbf{Rep}_{\mathbb{C}}(P_o)$.

(b) From (a) we see that, in particular, every complex representation (V, ξ) of G^c defines a G^c-vector bundle on $\check{X}$, and hence an automorphic vector bundle $\mathcal{V}(\xi)$. There is a local system $V(\xi)$ of $\mathbb{C}$-vector spaces underlying $\mathcal{V}(\xi)$, which can be described as follows: for K sufficiently small, the fundamental group of $\Gamma_g \backslash X^+$ is the image Γ_g^c of $gKg^{-1} \cap G(\mathbb{Q})_+$ in $G^c(\mathbb{Q})_+$ (notation as in II.2); the restriction of $V_K(\xi)$ to $\Gamma_g \backslash X^+$ is defined by the representation of Γ_g^c on V given by ξ. It follows from (II.3.1) that $\mathcal{V}(\xi)$ in this case has a natural flat connection and that it is algebraic. (Note that a representation (V, ξ) of G^c defined over a subfield L of $\mathbb{C}$ gives rise, in the same way, to an L-local system on $\mathrm{Sh}(G, X)$ contained in $\mathcal{V}(\xi)^{\nabla(\xi)}$.)

(c) There is an infinite-dimensional version of the above construction: $(\mathfrak{g}, P_o)$-modules (not necessarily finite-dimensional) correspond to $G_{\mathbb{C}}$-equivariant quasi-coherent $\mathcal{D}$-modules on $\check{X}$, and the same construction as above defines a functor from the category of $G^c_{\mathbb{C}}$-equivariant quasi-coherent $\mathcal{D}$-modules on $\check{X}$ to the category of $G(\mathbb{A}_f)$-equivariant quasi-coherent $\mathcal{D}$-modules on $\mathrm{Sh}(G, X)$. Recall that a $(\mathfrak{g}, P_o)$-module is a P_o-module with an action of $\mathfrak{g}$ whose restriction to $\mathfrak{p}_o$ coincides with the differential of the P_o-action. In the case that the module is finite-dimensional, the action of $\mathfrak{g}$ can be integrated to an action of G extending that of P, and the corresponding D-module is coherent; it is therefore locally free (Borel et al. 1987, p211), and the $\mathcal{D}$-module structure on the module corresponds to a flat connection. Thus this case reverts to that discussed in (b).

Example 2.4. Let (G, X) be the pair, as in (II.2.4), associated with a symplectic space (V, ψ). There is a naturally defined abelian scheme $\mathcal{A}$ over $\mathrm{Sh}(G, X)$ (cf. II.3.11). A point $o \in \check{X}$ corresponds to a maximal isotropic subspace W of $V(\mathbb{C})$, and P_o is the subgroup of G stabilizing W. Write S for $\mathrm{Sh}(G, X)$, and $2g$ for the dimension of V.

(a) The automorphic vector bundle associated with the natural representation of P_o on V/W is the tangent space of $\mathcal{A}/S$.

(b) The line bundle $\omega(\mathcal{A}/S)$ is the dual of the automorphic vector bundle associated with the determinant of the natural representation of P_o on V/W.

(c) The canonical line bundle on S is the automorphic vector bundle associated with the $(g+1)^{\mathrm{st}}$-power of the determinant of the natural representation of P_o on V/W.

(d) The automorphic vector bundle associated with the standard representation of G on V is $\mathcal{H}_{\mathrm{dR}}(\mathcal{A})$, and the flat connection on it is the Gauss-Manin connection.

Relation to automorphic forms in the classical sense. The above discussion also makes sense for connected Shimura varieties $\mathrm{Sh}^0(G, X^+)$: β defines an embedding $X^+ \hookrightarrow \check{X}$, and a $G_{\mathbb{C}}$-vector bundle $\mathcal{J}$ on $\check{X}$ defines an *automorphic vector bundle* $\mathcal{V}^0(\mathcal{J})$ on $\mathrm{Sh}^0(G, X^+)$. We now explain how to interprete sections of such bundles as holomorphic automorphic forms in the classical sense.

Let Γ be a discrete subgroup of $\mathrm{Aut}(X^+)$. Classically, one defines an *automorphy factor* for (Γ, X^+) with values in a complex vector space V to be a mapping $j : \Gamma \times X^+ \to GL(V)$ such that:

(a) for each $\gamma \in \Gamma$, $x \mapsto j(\gamma, x)$ is holomorphic on X^+;

(b) $j(\gamma\gamma', x) = j(\gamma, \gamma' x) \cdot j(\gamma', x)$, all γ, $\gamma' \in \Gamma$, $x \in X^+$.

An *automorphic form* for Γ of type j is then a function $f : X^+ \to V$ such that

(a) f is holomorphic;

(b) $f(\gamma x) = j(\gamma, x) f(x)$;

(c) f is "holomorphic at infinity".

Let $\mathcal{J}$ be a $G_{\mathbb{C}}$-vector bundle on $\check{X}$; choose a point $o \in X^+$, and let $V = \mathcal{J}_{\beta(o)}$. Because X^+ is simply connected, the isomorphism $V \to \beta^{-1}(\mathcal{J})_o$ extends to an isomorphism $X^+ \times V \approx \beta^{-1}(\mathcal{J})$, and we can transfer the action of $G(\mathbb{R})^+$ on $\beta^{-1}(\mathcal{J})$ to $X^+ \times V$. Write

$$\gamma(x, v) = (\gamma x, j(\gamma, x) v) \text{ for } \gamma \in G(\mathbb{R})^+,\ x \in X^+, \text{ and } v \in V.$$

Then $j : G(\mathbb{R})^+ \times V \to V$ satisfies the conditions (a) and (b), and so its restriction to $\Gamma_K \times V$ is an automorphy factor. A section of $\mathcal{V}^0(\mathcal{J})_K$ on $\mathrm{Sh}^0_K(G, X^+)$ can then be identified with an automorphic form for Γ_K of type j.

Example 2.5. Let $G = SL_2$, and let X^+ be the complex upper-half-plane (see II.1.5). The map $z \mapsto \frac{z-\mathrm{i}}{z+\mathrm{i}}$ is an isomorphism from X^+ to $D = \{z \in \mathbb{C} \mid |z| < 1\}$. In this case $\check{X}$ is the Riemann sphere, and $X \hookrightarrow \check{X}$ is an isomorphism of X with the upper hemisphere. If we take $o = \mathrm{i}$ (in the upper-half-plane), then $P_0 = \left\{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right\}$. If χ_k is the $2k^{\text{th}}$ power of the obvious character of P_0 and $\mathcal{V}_k$ is the corresponding automorphic vector bundle, then the sections of $\mathcal{V}_k$ holomorphic at infinity are elliptic modular forms of weight k.

3. The standard principal bundle.

The functor $\mathcal{J} \mapsto \mathcal{V}(\mathcal{J})$ takes one algebraic object to a second, but passes through the intermediary of the non-algebraic object X. In order to understand the rationality properties of the functor, we need to replace X by an algebraic object — this we call the standard principal bundle.

Review of principal bundles. Let S a complex manifold, and let G be a complex Lie group. A *flat structure* on a principal G-bundle P is given by a covering U_α of S for which the transition maps are constant.

Assume S is connected, and let $\tilde{S}$ be the universal covering space of S. A homomorphism $\xi : \pi_1(S, s) \to G$ defines a principal G-bundle

$$P(\xi) = \tilde{S} \times G/\sim, \; (s\gamma, g) \sim (s, \xi(\gamma)g), \;\; s \in \tilde{S}, \; \gamma \in \pi_1(S, s), \; g \in G,$$

on S, and there is a canonical flat structure on $P(\xi)$. Every principal G-bundle P over S admitting a flat structure arises in this way. In the case that $G = GL(V)$, V a $\mathbb{C}$-vector space, $P(\xi)$ is the frame bundle of $\mathcal{V}(\xi)$: the sections of $P(\xi)$ over an open subset U of S can be identified with the isomorphisms $a : U \times V \xrightarrow{\approx} \mathcal{V}|U$ (trivializations of $\mathcal{V}$ over U). Now suppose that ξ factors through a reductive algebraic subgroup G of $GL(V)$. Then $P(\xi)$ can be interpreted as the bundle of frames of $\mathcal{V}(\xi)$ respecting certain tensors. When S is a complex algebraic variety and $\mathcal{V}(\xi)$ and the tensors are algebraic, then $P(\xi)$ is also algebraic: it is a G-torsor over S.

LEMMA 3.1. *Let G be an algebraic group over a field k, and let $\pi : P \to S$ be a torsor for G over an algebraic k-variety S.*

(a) The functor $\mathcal{V} \mapsto \pi^{-1}\mathcal{V}$ defines an equivalence between the category of vector bundles on S and the category of G-vector bundles on P.

(b) If P has a flat structure, then to give a (flat) connection on $\mathcal{V}$ is the same as to give a (flat) connection in $\pi^{-1}(\mathcal{V})$.

PROOF: This is a standard consequence of descent theory.

Define

$$P(G,X) = G(\mathbb{Q})\backslash X \times G^c(\mathbb{C}) \times G(\mathbb{A}_f)/Z(\mathbb{Q})^-,$$

$$q(x,c,a)z = (qx, qc, qaz),\ q \in G(\mathbb{Q}),\ z \in Z(\mathbb{Q})^-.$$

Then $P(G,X)$ is a principal $G^c(\mathbb{C})$-bundle on $\mathrm{Sh}(G,X)^{\mathrm{an}}$, which we call the *standard principal bundle*. The group $G(\mathbb{A}') =_{\mathrm{df}} G(\mathbb{C}) \times G(\mathbb{A}_f)$ acts on $P(G,X)$ according to the rule

$$[x,z,a](c,g) = [x, zc, ag],\ x \in X,\ z,c \in G(\mathbb{C}),\ a,g \in G(\mathbb{A}_f).$$

Write π for the projection map $P(G,X) \to \mathrm{Sh}(G,X)$.

PROPOSITION 3.2. *The bundle $P(G,X)$ is algebraic, and the action of $G(\mathbb{A}')$ is algebraic.*

PROOF: For any faithful representation (V,ξ) of $G^c_{\mathbb{C}}$, $P(G,X)$ is the bundle of frames, respecting certain tensors, of the vector bundle $\mathcal{V}(\xi)$. Now apply (II.3.1).

Remark 3.3. Let ξ be as in the above proof. The functor represented by $P(G,X)$ can be described as follows: for any morphism $\gamma : T \to \mathrm{Sh}(G,X)$, the liftings of γ to $P(G,X)$ correspond to the trivializations $T \times V \to \gamma^{-1}(\mathcal{V}(\xi))$ of $\gamma^{-1}(\mathcal{V}(\xi))$ respecting certain tensors.

For example, suppose (G,X) satisfies (II.2.1*) and is of Hodge type. Corresponding to a symplectic representation $\xi : G \hookrightarrow GSp(V,\psi)$ there is an abelian scheme $\mathcal{A}$ over $\mathrm{Sh}(G,X)$ such that $\mathcal{H}_B(\mathcal{A}) = V(\xi)$. For any point $s \in \mathrm{Sh}(G,X)$, $\pi^{-1}(s)$ is equal to the set of morphisms $H_B(\mathcal{A}_s) \otimes \mathbb{C} \to H_{\mathrm{dR}}(\mathcal{A}_s)$ respecting certain Hodge cycles on $\mathcal{A}_s$.

PROPOSITION 3.4. *There is a canonical $G(\mathbb{C})$-equivariant map $\gamma : P(G,X) \to \check{X}$.*

PROOF: Choose a faithful representation $\xi : G^c_{\mathbb{C}} \hookrightarrow GL(V)$, as before. The last remark shows that a complex point p of $P(G,X)$ corresponds to an isomorphism $V \to \mathcal{V}(\xi)_{\pi(p)}$ respecting certain tensors. The Hodge filtration on $\mathcal{V}(\xi)_{\pi(p)}$ pulls back to a filtration on V, and we can map p to the corresponding point of $\check{X}$. That this is a morphism of algebraic varieties follows from the universal property of $\check{X}$ described in §1.

PROPOSITION 3.5. *Let $\mathcal{J}$ be a $G_{\mathbb{C}}$- vector bundle on $\check{X}$. Then $\mathcal{V}(\mathcal{J})$ is the unique vector bundle on* $\mathrm{Sh}(G, X)$ *such that $\pi^{-1}(\mathcal{V}(\mathcal{J})) = \gamma^{-1}(\mathcal{J})$ (as a G^c-vector bundle).*

PROOF: This follows directly from (3.3) and the definitions.

The following diagram summarizes the situation:

$$\begin{tikzcd}
\mathcal{V}(\mathcal{J}) \arrow[r, dash] & \pi^{-1}(\mathcal{V}(\mathcal{J})) = \gamma^{-1}(\mathcal{J}) \arrow[r, dash] & \mathcal{J} \\
\mathrm{Sh}(G, X) & P(G, X) \arrow[l, "\pi"'] \arrow[r, "\gamma"] & \check{X} \\
& X \arrow[ul] \arrow[u] \arrow[ur, dashed] &
\end{tikzcd}$$

Remark 3.6. Proposition 3.5 provides an alternative proof that the vector bundles $\mathcal{V}(\mathcal{J})$ are algebraic.

4. Canonical models of standard principal bundles. The key result that allows us to construct canonical models is the following.

THEOREM 4.1. *Let $\tau \in Aut(\mathbb{C})$.*

(a) For any special point $x \in X$, $\varphi_{\tau,x}$ lifts canonically to an equivariant morphism

$$\varphi^P_{\tau,x} : \tau P(G, X) \to P({}^{\tau,x}G, {}^{\tau,x}X).$$

(b) If x' is a second special point, then $\varphi(\tau; x', x)$ lifts canonically to an equivariant morphism

$$\varphi^P(\tau; x', x) : P({}^{\tau,x}G, {}^{\tau,x}X) \to P({}^{\tau,x'}G, {}^{\tau,x'}X),$$

and

$$\varphi^P(\tau; x', x) \circ \varphi^P_{\tau,x} = \varphi^P_{\tau,x'}.$$

PROOF: The strategy is that outlined in (II.9); see the notes at the end of the Chapter.

Example 4.2. (a) Suppose that (G, X) is of Hodge type, and that it satisfies (II.2.1.2*). Then the choice of a faithful representation (V, ξ) of G defines an abelian scheme $\mathcal{A}$ (with additional structure) on $\mathrm{Sh}(G, X)$. From a CM-point x, we obtain a representation of $({}^{\tau,x}V, {}^{\tau,x}\xi)$ of ${}^{\tau,x}G$, and therefore an abelian scheme (with additional structure) ${}^{\tau,x}\mathcal{A}$ on $\mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$. Under our hypotheses, $\mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$ is a fine moduli variety and ${}^{\tau,x}\mathcal{A}$ is the universal abelian scheme over it. The universality implies the existence of a commutative diagram:

$$\begin{array}{ccc}
\tau\mathcal{A} & \longrightarrow & {}^{\tau,x}\mathcal{A} \\
\downarrow & & \downarrow \\
\tau\mathrm{Sh}(G,X) & \longrightarrow & \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X).
\end{array}$$

Since $\mathcal{V}(\xi) = \mathcal{H}_{\mathrm{dR}}(\mathcal{A})$ and $\mathcal{V}({}^{\tau,x}\xi) = \mathcal{H}_{\mathrm{dR}}({}^{\tau,x}\mathcal{A})$, and $P(G, X)$ and $P({}^{\tau,x}G, {}^{\tau,x}X)$ are the frame bundles of $V(\xi)$ and $\mathcal{V}({}^{\tau,x}\xi)$, the diagram gives $\varphi^P_{\tau,x}$.

(b) For the Shimura variety defined by a CM-pair (T, x), it is possible to give an explicit description of $\varphi^P_{\tau,x}$ in terms of the period torsor.

THEOREM 4.3. *(a) The standard principal bundle $P(G, X)$ has a canonical model $P(G, X)_E$ over $E = E(G, X)$.*

(b) For any $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, $\tau P(G, X)_E$ is a canonical model of $P({}^{\tau,x}G, {}^{\tau,x}X)$.

PROOF: This can be deduced from (4.1) in the same way as (II.5.5) is deduced from (II.4.2) and (II.4.4).

Example 4.4. (a) In the situation of (4.2a), $\mathcal{A}$ is defined over the canonical model $\mathrm{Sh}(G, X)_E$, and for any point $s \in \mathrm{Sh}(G, X)_E$, $\pi^{-1}(s)$ is equal to the set of morphisms $H_B(\mathcal{A}_s) \otimes E \to H_{\mathrm{dR}}(\mathcal{A}_s/E)$ respecting certain Hodge cycles on $\mathcal{A}_s$.

(b) In the situation of (4.2b), it is possible to give an explicit description of $P(T, x)_E$ in terms of the period torsor.

Remark 4.5. The following properties of $\varphi^P_{\tau,x}$ provide justification for calling it canonical.

(i) A morphism $(G, X) \to (G', X')$ and a special point $x \in X$ give rise to a commutative diagram,

$$\begin{array}{ccc} \varphi^P_{\tau,x} : \tau P(G,X) & \longrightarrow & P({}^{\tau,x}G, {}^{\tau,x}X) \\ \downarrow & & \downarrow \\ \varphi^P_{\tau,x} : \tau P(G',X') & \longrightarrow & P({}^{\tau,x'}G', {}^{\tau,x'}X). \end{array}$$

Here x' is the image of x in X'.

(ii) Consider two pairs (G,X) and (G',X') together with an identification $(G^{\mathrm{der}}, X^+) = (G'^{\mathrm{der}}, X'^+)$. Let x be a special point of X^+, and let x' be the corresponding point of X'^+. Then there is an equivariant commutative diagram:

$$\begin{array}{ccccc} \tau P(G,X) & \longleftarrow & \tau P^0(G^{\mathrm{der}}, X^+) & \longrightarrow & \tau P(G',X') \\ \downarrow{\scriptstyle \varphi^P_{\tau,x}} & & \downarrow & & \downarrow{\scriptstyle \varphi^P_{\tau,x'}} \\ P({}^{\tau,x}G, {}^{\tau,x}X) & \longleftarrow & P^0({}^{\tau,x}G^{\mathrm{der}}, {}^{\tau,x}X^+) & \longrightarrow & P({}^{\tau,x'}G', {}^{\tau,x'}X') \end{array}$$

where $P^0(G^{\mathrm{der}}, X^+)$ is a principal bundle for G^{der} on $\mathrm{Sh}^0(G^{\mathrm{der}}, X^+)$ and $P^0({}^{\tau,x}G^{\mathrm{der}}, {}^{\tau,x}X^+)$ is a certain principal bundle for ${}^{\tau,x}G^{\mathrm{der}}$ on $\mathrm{Sh}^0({}^{\tau,x}G^{\mathrm{der}}, {}^{\tau,x}X^+)$.

The family of maps $(\varphi^P_{\tau,x})$ is uniquely determined by the properties (i) and (ii) and that mentioned in (4.2b).

So far as the canonical model of $P(G,X)$ is concerned all one can say in general is that it is constructed in a canonical fashion using the (canonical) maps $\varphi^P_{\tau,x}$. However, if one is prepared to confine one's attention to Shimura varieties whose weight is defined over $\mathbb{Q}$, it is possible to give a characterization similar to that for canonical models of Shimura varieties: the map $P(G,X) \to P(G',X')$ defined by a morphism $(G,X) \to (G',X')$ is defined over any field containing the reflex fields of (G,X) and (G',X'); for a pair (T,x) as in (4.2b), there is an explicit description of the canonical model of $P(T,x)$ in terms of the period torsor; the canonical model of $P(G,X)$ is uniquely determined by the condition that, for each CM-pair $(T,x) \subset (G,X)$, $P(T,x) \to P(G,X)$ is defined over $E(T,x)$.

THEOREM 4.6. *(a) The map* $\gamma : P(G,X) \to \check{X}$ *is rational over* $E(G,X)$.

(b) For any $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, *the diagram*

$$\begin{tikzcd} \tau P(G,X)_E \arrow[r] \arrow[d] & \tau \check{X}_E \arrow[d] \\ P({}^{\tau,x}G, {}^{\tau,x}X)_{\tau E} \arrow[r] & {}^{\tau,x}X_{\tau E} \end{tikzcd}$$

commutes.

PROOF: See the notes at the end of the Chapter.

5. Canonical models of automorphic vector bundles.

From the results in §4 on the standard principal bundle, it is possible to read off similar results for automorphic bundles.

THEOREM 5.1. *Let* $\mathcal{J}$ *be a* G^c*-vector bundle on* $\check{X}$*, and assume that* $\mathcal{J}$ *is defined over a number field* $E \supset E(G,X)$.

(a) The automorphic vector bundle $\mathcal{V}(\mathcal{J})$ *has a canonical model* $\mathcal{V}(\mathcal{J})_E$ *over* E.

(b) Let τ *be an automorphism of* $\mathbb{C}$*, and let* ${}^{\tau,x}\mathcal{J}$ *be the vector bundle on* ${}^{\tau,x}\check{X}$ *corresponding to* $\tau\mathcal{J}$ *under the isomorphism of (1.3). There is a canonical commutative diagram*

$$\begin{tikzcd} \tau\mathcal{V}(\mathcal{J})_E \arrow[r] \arrow[d] & \mathcal{V}({}^{\tau,x}\mathcal{J})_{\tau E} \arrow[d] \\ \tau \mathrm{Sh}(G,X)_E \arrow[r] & \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)_{\tau E}; \end{tikzcd}$$

that is, $\tau\mathcal{V}(\mathcal{J})_E$ *is isomorphic to the canonical model of* $\mathcal{V}({}^{\tau,x}\mathcal{J})$.

When $\mathcal{J}$ *is defined by a representation* (V,ξ) *of* G^c*, then the flat connection* $\nabla(\xi)$ *descends to the canonical model* $\mathcal{V}(\xi)_E$*, and the isomorphism in (b) respects the flat connections on* $\tau\mathcal{V}(\mathcal{J})_E$ *and* $\mathcal{V}({}^{\tau,x}\mathcal{J})_{\tau E}$.

PROOF: According to (4.3) and (4.5), the maps

$$\mathrm{Sh}(G,X) \xleftarrow{\pi} P(G,X) \xrightarrow{\gamma} \check{X}$$

are defined over E. We define $\mathcal{V}(\mathcal{J})_E$ to be the vector bundle on $\mathrm{Sh}(G,X)_E$ such that $\pi^{-1}(\mathcal{V}(\mathcal{J})_E) = \gamma^{-1}(\mathcal{J}_E)$ (see 4.4). Part (b) can be proved using the diagram:

$$\begin{tikzcd} \tau\mathrm{Sh}(G,X)_E \arrow[d] & \tau P(G,X)_E \arrow[l] \arrow[r] \arrow[d] & \tau\check{X}_E \arrow[d] \\ \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)_{\tau E} & P({}^{\tau,x}G, {}^{\tau,x}X)_{\tau E} \arrow[l] \arrow[r] & {}^{\tau,x}X_{\tau E}. \end{tikzcd}$$

Remark 5.2. Any equivariant differential operator $D : \mathcal{J} \to \mathcal{J}'$ between $G_{\mathbb{C}}$-vector bundles on $\check{X}$ induces a differential operator $\mathcal{V}(D) : \mathcal{V}(\mathcal{J}) \to \mathcal{V}(\mathcal{J}')$ between the $G(\mathbb{A}_f)$-vector bundles on $\mathrm{Sh}(G, X)$. If D, $\mathcal{J}$, and $\mathcal{J}'$ are defined over $E \supset E(G, X)$, then so also is $\mathcal{V}(D)$.

Remark 5.3. It would be of interest to re-interprete the above results in the context of (II.6), and to extend (II.7.2) to the standard principal bundle.

6. The local systems defined by a rational representation. We examine in more detail the various local systems defined by a representation (V, ξ) of G^c. As is explained above and in Chapter II, attached to such a representation we have:

(a) a local system of $\mathbb{Q}$-vector spaces $V(\xi)$ on $\mathrm{Sh}(G, X)$;

(b) a local system of $\mathbb{Q}_\ell$-vector spaces $V_\ell(\xi)$ on $\mathrm{Sh}(G, X)$;

(c) a vector bundle $\mathcal{V}(\xi)$ with a flat connection $\nabla(\xi)$ on $\mathrm{Sh}(G, X)$.

These are related by canonical comparison isomorphisms:

(i) $V(\xi) \otimes \mathbb{Q}_\ell \to V_\ell(\xi)$;

(ii) $V(\xi) \otimes \mathbb{C} \to \mathcal{V}(\xi)^{\nabla(\xi)}$.

All these objects have an action of $G(\mathbb{A}_f)$, and the comparison isomorphisms are compatible with the actions.

Remark 6.1. It is an elementary result that $V_\ell(\xi)$ has a canonical model over $E(G, X)$. For K sufficiently small, $\mathrm{Sh}(G, X)$ is Galois over $\mathrm{Sh}_K(G, X)$ with Galois group the image K^c of K in $G^c(\mathbb{A}_f)$, and $V_\ell(\xi)$ is the sheaf on $\mathrm{Sh}_K(G, X)$ corresponding to the representation of K^c on $V \otimes \mathbb{Q}_\ell$ defined by ξ. This construction works over $E(G, X)$, and gives us the canonical model of $V_\ell(\xi)$. Moreover, when the weight w_X is defined over $\mathbb{Q}$, the local system $\tau V_\ell(\xi)$ on $\tau\mathrm{Sh}(G, X)$ corresponds under $\varphi_{\tau,x}$ to $V_\ell({}^{\tau,x}\xi)$, where ${}^{\tau,x}\xi$ is the representation of ${}^{\tau,x}G$ obtained from ξ by twisting by ${}^{\tau}\mathfrak{S}$.

The objects in (b) and (c) are algebraic, and we can think of $V(\xi)$ as providing a rational structure to the family $(\mathcal{V}_\ell(\xi), (\mathcal{V}(\xi), \nabla(\xi))$. The next result shows that the family $(\tau V_\ell(\xi), \tau(\mathcal{V}(\xi), \nabla(\xi))$ on $\tau\mathrm{Sh}(G, X)$ also has a canonical rational structure.

THEOREM 6.2. *Let τ be an automorphism of $\mathbb{C}$, and let (V, ξ) be a representation of G^c. Assume that the composite of the weight map w_X with $G \to G^c$ is defined over $\mathbb{Q}$. Then there is a canonical local system ${}^{\tau}V(\xi)$ of $\mathbb{Q}$-vector spaces on $\tau\mathrm{Sh}(G, X)$ such that*

(a) ${}^{\tau}V(\xi) \otimes \mathbb{Q}_\ell = \tau V_\ell(\xi)$, for all primes ℓ;

(b) ${}^{\tau}V(\xi) \otimes \mathbb{C} = (\tau\mathcal{V}(\xi))^{\tau\nabla(\xi)}$.

PROOF: We can use ${}^{\tau}\mathfrak{S}$ and the map $\rho_x : \mathfrak{S} \to G^c$ to twist the representation (V, ξ), and so obtain a representation $({}^{\tau,x}V, {}^{\tau,x}\xi)$ of ${}^{\tau,x}G^c$. Define ${}^{\tau}V(\xi)$ to be the local system of $\mathbb{Q}$-vector spaces on $\tau\mathrm{Sh}(G,X)$ corresponding to $V({}^{\tau,x}\xi)$ under the isomorphism $\varphi_{\tau,x}$. Theorem 4.4 implies that ${}^{\tau}V(\xi)$ is independent of the choice of x, and it follows directly from its construction that ${}^{\tau}V$ satisfies (a) and (b).

If we assume (II.3.9, 3.10), then (V,ξ) defines a family of motives $\mathcal{M}$ on $\mathrm{Sh}(G,X)$, and we should have

${}^{\tau}V(\xi) = \mathcal{H}_B(\tau\mathcal{M}) \quad (\tau\mathcal{M}$ on $\tau\mathrm{Sh}(G,X))$;

$V_\ell(\xi)_E = \mathcal{H}_\ell(\mathcal{M}_E)$, $\mathcal{M}_E$ the canonical model of $\mathcal{M}$ over $\mathrm{Sh}(G,X)_E$;

$(\mathcal{V}(\xi), \nabla(\xi))_E = \mathcal{H}_{\mathrm{dR}}(M_E)$ with its Gauss-Manin connection.

7. Automorphic forms rational over a subfield of $\mathbb{C}$.

Definition 7.1. Let $\mathcal{J}$ be a $G_{\mathbb{C}}$-vector bundle on $\check{X}$, rational over a number field E, with $E(G,X) \subset E \subset \mathbb{C}$. An automorphic form f of type $\mathcal{J}$ and level K is *rational over* E if it arises from a section of $\mathcal{V}_K(\mathcal{J})_E$ over $\mathrm{Sh}_K(G,X)_E$.

Write $A_K(\mathcal{J})_E = A_K(G,X,\mathcal{J})_E$ for the space of such forms; it is a vector space over E.

PROPOSITION 7.2. *With the above notations:*

$$A_K(\mathcal{J})_E \otimes_E \mathbb{C} = A_K(\mathcal{J})_{\mathbb{C}}.$$

PROOF: In general, if $\mathcal{V}$ is a vector bundle on a variety S over a field E, and C is an extension field of E, then $\Gamma(S,\mathcal{V}) \otimes_E C = \Gamma(S_C, \mathcal{V}_C)$.

COROLLARY 7.3. *The vector space $A_k(\mathcal{J})_E$ is finite-dimensional over E.*

PROOF: This follows from (2.1b).

We now discuss rationality criteria in terms of special values. Assume that the weight w_X is defined over $\mathbb{Q}$ and that (G,X) satisfies (II.2.1*). Consider the automorphic vector bundle $\mathcal{V}(\xi)$ defined by a representation (V,ξ) of G. For each CM-pair $(T,x) \subset (G,X)$, there is a unique homomorphism $\rho_x : \mathfrak{S} \to T$ such that $\mu_{\mathrm{can}} \circ \rho_x = \mu_x$ (see II.2.4). From the representation $(\xi|T) \circ \rho_x$ we obtain a CM-motive M over $\mathbb{Q}^{\mathrm{al}}$ with $H_B(M) = V$, and from the model M_E of M over the canonical model of $\mathrm{Sh}(T,x)$, we obtain an $E(T,x)$-structure $V_{E,x} =_{\mathrm{df}} H_{\mathrm{dR}}(M_E)$ on $V(\mathbb{C})$. It is also possible to construct $V_{E,x}$

directly from the period torsor. There is a canonical identification of $V_{E,x}$ with the fibre $\mathcal{V}(\xi)_{E,x}$. Thus, if an automorphic form f is defined over $E(G,X)$, then $f(x)$, regarded as an element of $\mathcal{V}(\xi)_x = V(\mathbb{C})$ lies in the subspace $V_{E,x}$; conversely, when this condition holds for all CM-points, then f is defined over $E(G,X)$.

8. Automorphic stacks. Throughout this chapter we have insisted that the action of $G_{\mathbb{C}}$ on a vector bundle $\mathcal{J}$ on $\check{X}$ factor through $G^c_{\mathbb{C}}$, and that a representation ξ of G factor through G^c . We now explain why we have made these assumptions, and why it would be better to avoid them. Then we explain how to do this.

Consider the case of a representation $\xi : G \to GL(V)$, and let K be a compact open subgroup of $G(\mathbb{A}_f)$. The connected components of $\mathrm{Sh}_K(G,X)$ are of the form $\Gamma_g\backslash X^+$ where Γ_g is the image of $\Gamma'_g =_{\mathrm{df}} gKg^{-1} \cap G(\mathbb{Q})_+$ in $G^{\mathrm{ad}}(\mathbb{Q})^+$; here $g \in G(\mathbb{A}_f)$ and X^+ is a connected component of X. When ξ factors through G^c we define $\mathcal{V}(\xi)$ to be the vector bundle whose restriction to $\Gamma_g\backslash X^+$ is $\Gamma^c_g\backslash X^+ \times V(\mathbb{C})$ where Γ^c_g is the image of Γ'_g in $G^c(\mathbb{Q})$. This makes sense because, when K is is sufficiently small, the map $\Gamma^c_g \to \Gamma_g$ is an isomorphism, the fibre of $\Gamma^c_g\backslash X^+ \times V(\mathbb{C}) \to \Gamma_g\backslash X^+$ over any point is isomorphic to $V(\mathbb{C})$, and $\mathcal{V}(\xi)$ is a vector bundle. When we drop this condition, $\mathcal{V}(\xi)$ will no longer be a vector bundle. Consider for example the pair (G,X) in (II.2.4b) defining the Hilbert modular variety, and assume $F \neq \mathbb{Q}$. The centre Z of G is $F^\times$. For $g = 1$, the kernel of $\Gamma'_g \to \Gamma_g$ is $K \cap Z(\mathbb{Q})$, which is equal to the set of elements of $F^\times$ congruent to 1 modulo some ideal. This will be of finite index in the group of units of $F^\times$, and so is never trivial. The fibre of $\Gamma'_g\backslash X \times V(\mathbb{C}) \to \Gamma_g\backslash X$ will be the quotient of $V(\mathbb{C})$ by the action of this kernel, and so we do not get a vector bundle by this construction. This same problem also occurs when trying to define the universal family of abelian varieties over $\mathrm{Sh}(G,X)$ (van der Geer 1988, Chapter X).

So why not simply do as have done in this chapter and exclude this them? Classically, one defines automorphic forms as functions on the universal covering space X transforming in certain ways relative to the group Γ. The reason we wish to consider them as sections of a vector bundle on $\mathrm{Sh}(G,X)$ is so that we can apply the methods of algebraic geometry. From the classical point of view, it is unnatural to exclude them.

So how do we handle them? Just as in the case of the universal abelian scheme over the Hilbert moduli variety, we should use stacks.

Briefly, the idea is to pass to a partial quotient of X which makes sense algebraically, and on which $\mathcal{V}(\xi)$ is an equivariant vector bundle. In this way we obtain the notion of an *automorphic stack.*

In the case that weight is defined over $\mathbb{Q}$, it is possible to consider a concrete realization of the stack. Let G' be the smallest subgroup of G such that all h factor through $G'_{\mathbb{R}}$. Then $Z_s(G') = 0$. Consider

$$\mathrm{Sh}'(G,X) =_{\mathrm{df}} G'(\mathbb{Q})\backslash X \times G(\mathbb{A}_f) = \varprojlim G'(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K.$$

It is a covering of $\mathrm{Sh}(G,X)$, and every $G_{\mathbb{C}}$-vector bundle on $\check{X}$ defines a vector bundle on $\mathrm{Sh}'(G,X)$.

All results in the chapter continue to hold *mutandis mutatis* for automorphic stacks. In fact, since the proofs proceed via connected Shimura varieties, where this problem doesn't arise, there is little extra difficulty in working with stacks rather than vector bundles.

Notes. The principal theme of this chapter has been the problem of making sense of what it means for an automorphic form to be defined over a number field. In the case of elliptic modular functions, there is no difficulty: a modular form is defined over a number field if and only if its Fourier coefficients lie in the field. Unfortunately, in higher dimension, Fourier-Jacobi series are much more difficult to work with (see Chapter VII); moreover this method can only apply to noncompact Shimura varieties.

There are basically four approaches to defining rationality of automorphic forms:

(a) using Fourier-Jacobi series (or their null-values...)

(b) in terms of the special values of the forms (that is, values at the special points);

(c) pulling-back to sub-Shimura varieties of type A_1;

(d) directly defining a canonical model of the automorphic vector bundle.

Of course these approaches are not independent, and all should give the same answer when they apply.

Shimura used special values (and periods) to define the notion of an automorphic form being rational over $\mathbb{Q}^{al}$—see Shimura (1979). For applications of his results, see Shimura (1980), (1981). He studies Fourier-Jacobi series in Shimura (1978a), (1978b). For certain Shimura varieties Garrett (1983) shows that (a), (b), and (c) lead to consistent notions of rationality.

Under the hypothesis that the weight w_X is defined over $\mathbb{Q}$ and (G, X) satisfies (2.1.2*), Harris (1985) defined a functor $\mathcal{J} \mapsto \mathcal{V}(\mathcal{J})_E$ from $G_{\mathbb{C}}$-vector bundles on $\check{X}$ to vector bundles on $\mathrm{Sh}(G, X)_E$, but did not show that the functor was canonical. This result was the inspiration for Milne (1988), which proves the major statements in this section in the context of connected Shimura varieties. They can be extended to (nonconnected) Shimura varieties by "induction" (in the sense of (II.10)). Full details will be given in the book mentioned in the introduction. See also Harris (1986) where the relation between (a) and (d) is investigated.

IV. ONE-MOTIVES

A mixed Hodge structure on a vector space is an increasing filtration of the vector space together with a Hodge structure on each of the quotients. Hodge structures degenerate into mixed Hodge structures. The cohomology groups of a complex algebraic variety (not necessarily smooth or complete) carry mixed Hodge structures.

Just as abelian varieties provide an algebro-geometric realization of certain Hodge structures, one-motives provide an algebro-geometric realization of certain mixed Hodge structures.

1. Mixed Hodge structures. A *mixed Hodge structure* is

(a) a finite-dimensional vector space V over $\mathbb{Q}$,

(b) a finite increasing (weight) filtration $W.$ on V, and

(c) a finite decreasing (Hodge) filtration $F^{\cdot}$ on $V \otimes \mathbb{C}$ such that, for each n, $F^{\cdot}$ induces a Hodge structure of weight n on

$$\mathrm{Gr}_n^W(V) =_{\mathrm{df}} W_n V / W_{n-1} V.$$

When $\mathbb{Q}$ in the definition is replaced by $k \subset \mathbb{R}$, we obtain the notion of a *mixed k-Hodge structure.*

Example 1.1. (a) A Hodge structure $(V, F^{\cdot})$ of weight n can be made into a mixed Hodge structure by setting $W_n V = V$ and $W_{n-1} V = 0$.

(b) The cohomology groups $H^n(X, \mathbb{Q})$ of any variety X over $\mathbb{C}$ (not necessarily nonsingular or complete) have natural mixed Hodge structures. This is the main theorem in Deligne (1975).

(c) Let (V, ψ) be a symplectic space over $\mathbb{Q}$, and endow $V \otimes \mathbb{R}$ with a Hodge structure of type $\{(-1,0), (0,-1)\}$ for which ψ is a Riemann form (i.e., such that $(2\pi i)\psi$ is a polarization of the Hodge structure). Write $F^{\cdot}$ for the corresponding filtration of $V \otimes \mathbb{C}$. Let W be a totally isotropic subspace of V, and let $W^{\perp}$ be the orthogonal complement of W in V. Then we have a filtration

$$\begin{array}{ccccccc} 0 & \subset & W & \subset & W^{\perp} & \subset & V \\ \| & & \| & & \| & & \| \\ W_{-3}V & & W_{-2}V & & W_{-1}V & & W_0V, \end{array}$$

and one can check that $(V, W_., F^{\cdot})$ is a mixed Hodge structure (see Brylinski 1983, 4.2.1).

The *level* of a mixed Hodge structure is the length of the shortest interval $[c, d]$ such that $F^p/F^{p+1} \neq 0 \Rightarrow c \leq p \leq d$. A *morphism* of mixed Hodge structures $F : V \to V'$ is a linear map $V \to V'$ respecting the weight filtrations on V and V' and the Hodge filtrations on $V \otimes \mathbb{C}$ and $V' \otimes \mathbb{C}$. The category of mixed Hodge structures has a natural structure of a Tannakian category. The *Mumford-Tate group* $MT(V)$ of a mixed Hodge structure V is defined to be the affine group scheme attached to the sub-Tannakian category generated by V and $\mathbb{Q}(1)$.

The canonical bigrading. Let V be a mixed Hodge structure. For integers p and q, set $\tilde{V}^{p,q}$ equal to

$$(W_n(V) \cap F^p(V)) \cap (W_n(V) \cap \bar{F}^q(V) + \sum_{2 \leq i} (W_{n-i}(V) \cap \bar{F}^{q-i+1}(V)),$$

where $n = -p - q$.

Then

(a) $V = \oplus_{p,q} \tilde{V}^{p,q}$;

(b) the projection of $W_n(V)$ onto $\mathrm{Gr}_n^W(V)$ induces an isomorphism

$$\tilde{V}^{p,q} \to H^{p,q}(\mathrm{Gr}_n^W(V))$$

for all p, q with $p + q = n$;

(c) $W_n(V) = \sum_{p+q \leq n} \tilde{V}^{p,q}$;

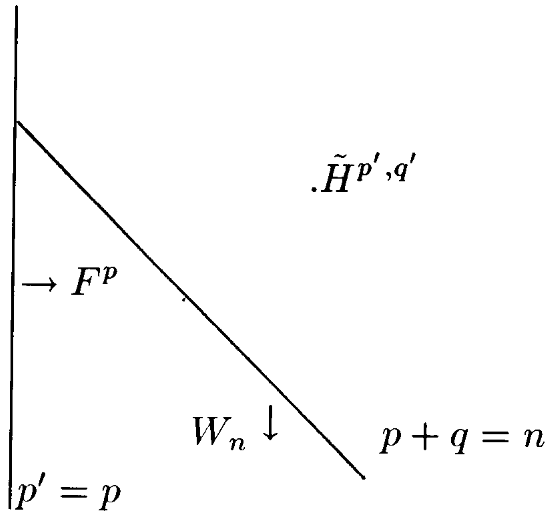
(d) $F^p(V) = \sum_{p' \geq p} \tilde{V}^{p',q}$.

(e) If W is a second mixed Hodge structure, then

$$(V \tilde{\otimes} W)^{m,n} = \otimes_{\substack{p+p'=m \\ q+q'=n}} \tilde{V}^{p,q} \otimes \tilde{W}^{p',q'}.$$

(f) A morphism of mixed Hodge structures respects the bigrading.

For the proof, see Deligne (1971a), 1.2.10, 1.2.11. We may visualize (c) and (d) as:



Clearly, an element of $V(\mathbb{R})$ is in $\tilde{V}^{0,0}$ if and and only if it is in both W_0V and F^0V. An element of a space $T = V^{\otimes m} \otimes \check{V}^{\otimes n} \otimes \mathbb{Q}(r)$ lying in $\tilde{H}^{0,0}$ (or a sum of such elements) will be called a *Hodge tensor* of V. As before, we let $\mathbb{G}_m$ act on T through its action on $\mathbb{Q}(1)$. Define

$$\tilde{h} : \mathbb{S}_{\mathbb{C}} \to GL(V(\mathbb{C})), \quad \tilde{h}(z_1, z_2) \cdot v = z_1^{-p} z_2^{-q} \cdot v, \; v \in \tilde{V}^{p,q},$$

and define $\tilde{h}' : \mathbb{S}_{\mathbb{C}} \to GL(V(\mathbb{C})) \times \mathbb{G}_m$ to be $(z_1, z_2) \mapsto (\tilde{h}(z_1, z_2), z_1 z_2)$. Then $t \in T$ is a Hodge tensor if and only if it is fixed by $\operatorname{Im}(\tilde{h}')$.

PROPOSITION 1.2. *(a) The Mumford-Tate group of V is the subgroup of $GL(V) \times \mathbb{G}_m$ of elements fixing all Hodge tensors of V.*

(b) The Mumford-Tate group of V is the smallest subgroup of $GL(V) \times \mathbb{G}_m$ whose complex points contain the image of $\tilde{h}'$.

PROOF: (a) With any $t \in V^{\otimes m} \otimes \check{V}^{\otimes n} \otimes \mathbb{Q}(r)$ we can associate an $\alpha(t) \in \operatorname{Hom}(V^{\otimes n}, V^{\otimes m}(r))$, and t will be a Hodge cycle if and only if $\alpha(t)$ is a morphism of Hodge structures. From this it follows that Hodge tensors are fixed by $GL(V) \times \mathbb{G}_m$, and that $\operatorname{Im}(\tilde{h}') \subset MT(V)(\mathbb{C})$. Thus the tensors fixed by $MT(V)$ are precisely the Hodge tensors.

Let M' be the subgroup of $GL(V) \times \mathbb{G}_m$ fixing the Hodge tensors. According to Deligne (1982a), 3.1c, in order to prove that $M' = MT(V)$, it suffices to show that every $\mathbb{Q}$-rational character of $MT(V)$ extends to $GL(V) \times \mathbb{G}_m$. Let $\chi : MT(V) \to GL(W)$ be such a character. Then W acquires a mixed Hodge structure, and since it has dimension one, we must have $W \approx \mathbb{Q}(r)$ for some r. It is now obvious that χ extends to $GL(V) \times \mathbb{G}_m$.

(b) Let H be the smallest subgroup of $GL(V) \times \mathbb{G}_m$ such that $H(\mathbb{C})$ contains the image of $\tilde{h}'$. Then an element of some subquotient S of $V^{\otimes m} \otimes \check{V}^{\otimes n} \otimes \mathbb{Q}(r)$ is in $\tilde{S}^{0,0}$ if and only if it is fixed by H. Thus $MT(V)$ and H fix the same tensors in all such subquotients, and this shows that the two groups are equal (see Deligne (1982a), 3.2a).

PROPOSITION 1.3. *Let G be an algebraic group over $\mathbb{R}$, and let $W.$ and $F^{\cdot}$ be filtrations of* $\mathbf{Rep}(G)$. *Suppose that for some family (V_i, ξ_i) of representations of G such that $\cap \mathrm{Ker}(\xi_i)$ is finite, $(W., F^{\cdot})$ defines a mixed Hodge structure on V_i for all i; then $(W., V^{\cdot})$ defines a mixed Hodge structure on V for all representations (V, ξ) of G.*

PROOF: See Deligne (1973), III.1.11.

Variations of mixed Hodge structures. A *variation of mixed Hodge structures* on a complex manifold S is

(a) a local system of $\mathbb{Q}$-vector spaces V on S,

(b) a filtration $W.$ of V by local systems $W_i V$.

(c) a holomorphic filtration $F^{\cdot}$ of $\mathcal{V} =_{\mathrm{df}} \mathcal{O}_S \otimes V$

such that

(H_1) $\nabla(F^p\mathcal{V}) \subset \Omega^1 \otimes F^{p-1}\mathcal{V}$

(H_2) for all $s \in S$, $(V_s, W._s, F^{\cdot}_s)$ is a mixed Hodge structure.

When $\mathbb{Q}$ in the definition is replaced by $k \subset \mathbb{R}$, then we obtain the notion of a *variation of mixed k-Hodge structures.* The families of mixed Hodge structures arising naturally in algebraic geometry are variations of mixed Hodge structures.

Notes. Mixed Hodge structures were introduced by Deligne in order to be able to state the theorem quoted in (1.1b). See Deligne (1971a), (1971b), (1975).

2. One-motives.

A *semi-abelian variety* over a field k is an extension of an abelian variety by a torus:

$$0 \to T \to G \to A \to 0.$$

When k is algebraically closed, a character χ of T then defines (by pushout) an element of $\mathrm{Ext}^1(A, \mathbb{G}_m) = \check{A}(k)$; conversely, a homomorphism $X^*(T) \to \check{A}(k)$ defines an extension of A by T.

A *one-motive* M over an algebraically closed field k is a triple (G_M, X_M, u) comprising a semi-abelian variety G_M over k, a finitely generated torsion-free abelian group X_M, and a homomorphism $u : X_M \to G_M(k)$. The definition when k is not algebraically closed is the same except that X_M is a $\mathrm{Gal}(k^{\mathrm{al}}/k)$-module and u is required to be an equivariant homomorphism $X_M \to G_M(k^{\mathrm{al}})$. We often drop the subscripts M, and write $M = (X \xrightarrow{u} G)$. We regard it as a complex of length one. Thus a *morphism* of one-motives is a commutative square:

$$\begin{array}{ccc} X & \xrightarrow{u} & G \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ X' & \xrightarrow{u'} & G'. \end{array}$$

A morphism (α,β) is an *isogeny* if the cokernel of α and the kernel of β are both finite. A one-motive has a filtration:

$$\begin{array}{rcccl} W_0M = & (X & \longrightarrow & G) & \\ & \cup & & \cup & \mathrm{Gr}_0(M) = X \\ W_{-1}M = & (0 & \longrightarrow & G) & \\ & \cup & & \cup & \mathrm{Gr}_{-1}(M) = A \\ W_{-2}M = & (0 & \longrightarrow & T) & \\ & \cup & & \cup & \mathrm{Gr}_{-2}(M) = T \\ & 0 & \longrightarrow & 0 & \end{array}$$

Betti homology. The Betti homology group of a one-motive M over $\mathbb{C}$ is a mixed Hodge structure $(H_B(M), F^{\cdot}, W_{\cdot})$ of type $\{(0,0); (0,-1), (-1,0); (-1,-1)\}$ such that

$$\mathrm{Gr}_0 H_B(M) = X \otimes \mathbb{Q}$$
$$\mathrm{Gr}_{-1} H_B(M) = H_1(A_M, \mathbb{Q})$$
$$\mathrm{Gr}_{-2} H_B(M) = H_1(T_M, \mathbb{Q}) \approx X_*(T) \otimes \mathbb{Q}.$$

To construct it, pull-back the top row of the following diagram by $X \to G$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(G) & \longrightarrow & \mathrm{Lie}(G) & \xrightarrow{\exp} & G & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H_1(G) & \longrightarrow & H_B(M,\mathbb{Z}) & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

and define $H_B(M) = H_B(M,\mathbb{Z}) \otimes \mathbb{Q}$.

THEOREM 2.1. *The functor $M \mapsto H_B(M)$ defines an equivalence between the category of one-motives over $\mathbb{C}$, considered up to isogeny, and the category of mixed Hodge structures of level ≤ 1 for which $Gr_{-1}H_B(M)$ is polarizable.*

PROOF: Deligne (1975), 10.1.3.

COROLLARY 2.2. *Let (V, h) be a Hodge structure of type $\{(-1,0), (0,-1)\}$, and let ψ be a polarization for (V, h). Let $W \subset V$ be a totally isotropic subspace. There is a unique one-motive M (up to isogeny) such that $H_B(M)$ is the mixed Hodge structure defined in (1.1c).*

Remark 2.3. The theorem explains the one in "one-motive". Note that one-motives are not motives but mixed motives (the Betti homology of a motive is a sum of (pure) Hodge structures).

The *Mumford-Tate group* MT^M of M is defined to be the Mumford-Tate group of the mixed Hodge structure $H_B(M)$.

de Rham homology. Let $M = (X \to G)$ be a one-motive over a field k. The exact sequence

$$0 \to X \to G \to M \to 0,$$

gives rise to an exact sequence of vector groups,

$$0 \to \mathrm{Hom}(X, \mathbb{G}_a) \to \mathrm{Ext}^1(G, \mathbb{G}_a) \to \mathrm{Ext}^1(M, \mathbb{G}_a) \to 0.$$

There is an extension $M^\natural = (X \to G^\natural)$ of M by $\mathrm{Ext}^1(M, \mathbb{G}_a)^\vee$, which fits into a diagram,

$$\begin{tikzcd} & & & X \arrow[d] \arrow[dl, dashed] & \\ 0 \arrow[r] & \mathrm{Ext}^1(M, \mathbb{G}_a)^\vee \arrow[r] & G^\natural \arrow[r] & G \arrow[r] & 0, \end{tikzcd}$$

and which is universal among extensions of M by vector groups (Deligne 1975, 10.1.7). Define $H_{\mathrm{dR}}(M) = \mathrm{Lie}(G^\natural)$. The map $M \mapsto H_{\mathrm{dR}}(M)$ is functorial in M, and so the weight filtration on M defines a filtration $W.$ on $H_{\mathrm{dR}}(M)$. The Hodge filtration is defined by

$F^{-1}H_{\mathrm{dR}}(M) = H_{\mathrm{dR}}(M)$,
$F^0 H_{\mathrm{dR}}(M) = \mathrm{Ext}^1(M, \mathbb{G}_a)^\vee = \mathrm{Ker}(\mathrm{Lie}\, G^\natural \to \mathrm{Lie}\, G)$,
$F^1 H_{\mathrm{dR}}(M) = 0$.

PROPOSITION 2.4. *When $k = \mathbb{C}$, there is a canonical isomorphism*

$$(H_{\mathrm{dR}}(M), F^{\cdot}, W_{\cdot}) \to (H_B(M) \otimes \mathbb{C}, F^{\cdot}, W_{\cdot}).$$

PROOF: See Deligne (1975), 10.1.8.

ℓ-adic homology. Let $M = (X \xrightarrow{u} G)$ be a one-motive over an algebraically closed field k, which, for simplicity, we take to be of characteristic zero. Define

$$M_m = H^0(M \otimes^{\mathbf{L}} (\mathbf{Z}/m\mathbf{Z})).$$

Thus M_m is the zero$^{\text{th}}$ cohomology group of the simple complex associated with the double complex:

$$\begin{array}{ccc} X & \xrightarrow{u} & G \\ \uparrow{\scriptstyle m} & & \uparrow{\scriptstyle -m} \\ X & \xrightarrow{u} & G, \end{array}$$

so that

$$M_m = \{(x,g) \in X \times G(k) \mid u(g) = mg\}/\{(mx, u(x)) \mid x \in X\}.$$

Define

$$H_\ell(M) = (\varprojlim M_{\ell^n}) \otimes_{\mathbf{Z}_\ell} \mathbb{Q}_\ell,$$

$$H_f(M) = {\prod}' H_\ell(M) \text{ (restricted product)}.$$

When k is not algebraically closed, we set $H_\ell(M) = H_\ell(M \otimes_k k^{\mathrm{al}})$.

PROPOSITION 2.5. *When $k = \mathbb{C}$ there is a canonical isomorphism $H_B(M) \otimes \mathbb{Q}_\ell \to H_\ell(M)$.*

PROOF: This amounts to checking that $H_B(M, \mathbf{Z}) \otimes (\mathbf{Z}/m\mathbb{Z}) = M_m$.

The dual one-motive. There is a functor sending a one-motive M to its dual $\check{M}$. Set $\check{X} = X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$,
$\check{A} =$ the dual abelian variety of A, $\mathrm{Ext}^1(A, \mathbb{G}_m)$,
$\check{T} = \mathrm{Hom}(X, \mathbb{G}_m)$.

Define $\check{G}$ to be $\mathrm{Ext}^1(M/W_{-2}M, \mathbb{G}_m)$. The sequence

$$0 \to X \to A \to M/W_{-2}M \to 0$$

gives rise to an exact sequence

$$0 \to \check{T} \to \check{G} \to \check{A} \to 0.$$

As M is an extension of $M/W_{-2}M$ by T, from each $x \in \check{X}$ we get an extension of $M/W_{-2}M$ by $\mathbb{G}_m$, and hence an element $\check{u}(k)$ of $\check{G}(k)$. This defines the map $\check{u}$, and completes the construction of $\check{M}$. There are the following formulas:

$H_B(\check{M}) = \mathrm{Hom}(H_B(M), \mathbb{Q}(1))$,
$H_\ell(\check{M}) = \mathrm{Hom}(H_\ell(M), \mathbb{Q}_\ell(1))$,
$H_{\mathrm{dR}}(\check{M}) = \mathrm{Hom}(H_{\mathrm{dR}}(M), k)$.

Symmetric one-motives. A *polarization* of a one-motive M is an isogeny $\lambda : M \to \check{M}$ such that $\mathrm{Gr}_{-1}(\lambda) : A \to \check{A}$ is a polarization of A. A one-motive together with a polarization, is called a *symmetric* one-motive.

PROPOSITION 2.6. *Giving a symmetric one-motive over k is equivalent to giving the following data:*

(a) a polarized abelian variety (A, λ) over k;

(b) a finitely generated torsion-free abelian group X with an action of $\mathrm{Gal}(k^{\mathrm{al}}/k)$;

(c) a $\mathrm{Gal}(k^{\mathrm{al}}/k)$-homorphism $v : X \to A(k^{\mathrm{al}})$; let $\check{v} = \lambda \circ v$;

(d) a trivialization ψ of the inverse image by $(v, \check{v})$ of the Poincaré biextension of A; ψ is required to be symmetric, i.e., invariant under $(x, x') \mapsto (x', x) : X \times X \to X \times X$.

PROOF: In fact, $(M, \lambda) \mapsto (\mathrm{Gr}_{-1}(M), \mathrm{Gr}_{-1}(\lambda), v)$ can be made into an equivalence of categories; cf. (Deligne 1975, 10.2.14).

We explain (d). The Poincaré line bundle is the line bundle on $A \times \check{A}$ which expresses the duality between A and $\check{A}$ (Mumford 1970, § 13). The Poincaré biextension is the $\mathbb{G}_m$ torsor on $A \times \check{A}$ obtained by removing the zero section from the Poincaré line bundle. Its inverse image by $(v, \check{v})$ is a $\mathbb{G}_m$-torsor L on $X \times X$ regarded as a scheme of dimension zero. If ψ is one trivialization, then any other is of the form $\psi \circ g$, with g an element of $\mathbb{G}_m(X \times X)$ invariant under the symmetry $X \times X \to X \times X$. Consequently, we have the following result.

COROLLARY 2.7. *The symmetric one-motives with $(A, \lambda, v : X \to A)$ fixed form a torsor under $Hom_{\mathrm{sym}}(X \times X, \mathbb{G}_m) = Hom(S^2(X), \mathbb{G}_m)$.*

Hodge cycles. When M is a one-motive over $\mathbb{C}$, we define a *Hodge cycle* on M to be a Hodge tensor for the mixed Hodge structure $H_B(M)$. Propositions 2.4 and 2.5 show that such a cycle has realizations in the de Rham and ℓ-adic homology groups of M. When M is defined over an algebraically closed field k, we say that a family $s = (s_{\mathrm{dR}}, (s_\ell))$ is a *Hodge cycle relative to an embedding* $\tau : k \hookrightarrow \mathbb{C}$ if the components of s become the components of a Hodge cycle s_0 on τM.

PROPOSITION 2.8. *Let M be a one-motive over an algebraically closed field k. If s is a Hodge cycle on M relative to one embedding of k in $\mathbb{C}$, then it is a Hodge cycle for every embedding.*

PROOF: The proof of (I.3.1) can be extended to one-motives; see Brylinski (1983), 2.2.5.

The procedure of (I.3) now allows us to define the notion of a Hodge cycle for a one-motive over any field of characteristic zero.

One-motives of CM-type. A one-motive $M = (X \xrightarrow{u} G)$ over a field k is said to be *rationally decomposed* if the image of u is finite and the class of G in $\mathrm{Ext}^1(A, T)$ is of finite order. It is then isogenous to the one motive $X \xrightarrow{0} T \times A$. When $k = \mathbb{C}$, M is rationally decomposed if and only if the mixed Hodge structure $H_B(M)$ is isomorphic to the direct sum of the pure Hodge structures $H_B(T)$, $H_B(A)$, and $X \otimes \mathbb{Q}$ (these are of types $\{(-1,-1)\}$, $\{(-1,0),(0,-1)\}$, and $\{(0,0)\}$ respectively. To such a one-motive M, we attach a motive

$$hM = h(X_*(T) \otimes \mathbb{Q}) \oplus h(A) \oplus h(X \otimes \mathbb{Q})$$

in $\mathbf{AV}/k$ (the first and last summands are elements of $\mathbf{Art}/k$; see I.4.1).

A one-motive M is said to be of *CM-type* if it is rationally decomposed and A_M is of CM-type. Then hM lies in $\mathbf{CM}/k$. In particular, when M is defined over $\mathbb{C}$, its Mumford-Tate group is a quotient of $\mathfrak{S}$, and when M is defined over $\mathbb{Q}$, it corresponds to a representation of the Taniyama group.

Moduli of one-motives. Let M be a one-motive over $\mathbb{C}$, and write $(H, W_., F^{\cdot})$ for $H_B(M)$ with its mixed Hodge structure. As in (I.1), the Mumford-Tate group P of M acquires a filtration

$$1 = W_{-3}P \subset W_{-2}P \subset W_{-1}P \subset W_0 P = P$$

from the weight filtration on H:

$$W_{-i}P = \{p \in P \mid (id - p)(W_m H_B(M)) \subset W_{m-i} H_B(M), \text{ all } m\}.$$

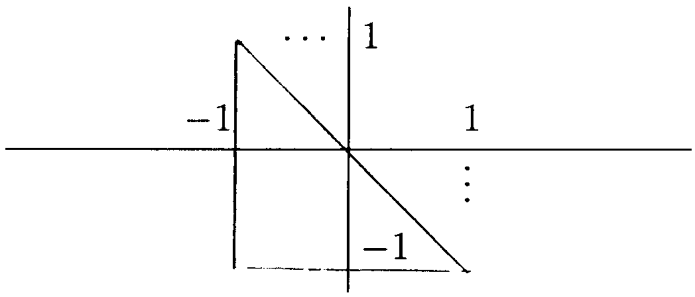
The group $W_{-1}P$ is unipotent, and the quotient $P/W_{-1}P$ is the Mumford-Tate group of $\mathrm{Gr}_{-1}(M) = A$. Therefore $P/W_{-1}P$ is reductive, and $W_{-1}P$ is the unipotent radical of P.

LEMMA 2.9. *(a) For all $p \in P(\mathbb{R}) \cdot W_{-1}P(\mathbb{C})$, the filtration $p \cdot F^{\cdot}$ of $H \otimes \mathbb{C}$ defines a mixed Hodge structure on (H, W).*

(b) There exists a $p \in W_{-1}P(\mathbb{C})$ such that the mixed Hodge structure $(H, W_{\cdot}, p \cdot F^{\cdot})$ is rationally decomposed.

PROOF: Brylinski (1983), 2.2.8 (see also VI.1).

LEMMA 2.10. *The mixed Hodge structure on Lie P defined by $(W_{\cdot}, p \cdot F^{\cdot})$ is of type $\{(-1,-1); (-1,0), (0,-1); (-1,1), (0,0), (1,-1)\}$.*



It follows that $F^0 P \cap W_{-1}P$ is commutative, because,

$$[F^0 P \cap W_{-1}P, F^0 P \cap W_{-1}P] \subset F^0 P \cap W_{-2}P = 0.$$

Choose a lattice $H(\mathbb{Z})$ in H. The family of one-motives $p \cdot M$, $p \in P(\mathbb{R}) \cdot W_{-1}P(\mathbb{C})$ is parametrized by the space

$$V = \Gamma \backslash P(\mathbb{R}) \cdot W_{-1}P(\mathbb{C}) / F^0 P(\mathbb{C})$$

where Γ is the subgroup of $P(\mathbb{Q})$ respecting the lattice.

THEOREM 2.11. *When Γ is replaced by a sufficiently small congruence subgroup, the variety V has a natural structure of an algebraic variety, and the analytic family of one-motives over it also has a natural structure of an algebraic variety.*

PROOF: Brylinski (1983), 2.3.2.1 (see also Chapter VI).

By introducing level structures and Hodge cycles, it is possible to strengthen the theorem in order to obtain a universal family of one-motives.

Notes. The concept of a one-motive is due to Deligne (1975).

3. Degenerating families of symmetric one-motives.

Understanding the boundaries of Shimura varieties of Hodge type is closely related to understanding the degeneration of abelian varieties and one-motives. The degeneration theorem we state below is an algebraic analogue of the following analytic statements. Let D be the unit disk and let $D' = D - \{0\}$. Consider functions $f_i : D \to \mathbb{C}$ such that $f_i(z) \neq 0$ for $z \neq 0$ and $f_i(0) = 0$ for $1 \leq i \leq r$. Let $\mathcal{T}$, $\mathcal{G}$, and $\mathcal{A}$ to be the complex manifolds over D whose fibres over $z \in D$ are:

$\mathcal{T}_z = \mathbb{C}^{\times r}$,
$\mathcal{G}_z = \mathbb{C}^{\times m} / \langle f_{r+1}(z), \dots, f_m(z) \rangle$,
$\mathcal{A}_z = \mathbb{C}^{\times m-r} / \langle f_{r+1}(z), \dots, f_m(z) \rangle$.

Here $\langle f_{r+1}(z), \dots, f_m(z) \rangle$ is the abelian subgroup generated by $f_{r+1}(z), \dots, f_m(z)$. There is an exact sequence

$$0 \to \mathcal{T} \to \mathcal{G} \to \mathcal{A} \to 1.$$

The functions $f_1, \dots, f_r$ define a map $u : X \to \mathcal{G}$ where X is the constant local system $\mathbb{Z}^r$ on D. Let $A = \mathcal{G}/u(X)$. Then A is the complex-analytic analogue of a semi-abelian variety, the map $\mathcal{G} \to A$ is a local isomorphism, and the fibre of A over 0 is equal to the fibre of $\mathcal{G}$ over 0.

Let R a Noetherian, excellent, normal ring that is complete with respect to a radical ideal $\mathfrak{I}$; let

$S = \operatorname{Spec} R$;
$\eta =$ generic point of $S = \operatorname{Spec} K$;
$S_0 = \operatorname{Spec} A/\mathfrak{I}$.

Intuitively, a degenerating one-motive over S is a one-motive over $S - S_0$ whose period group degnerates totally along S_0. It is most convenient to state the definition in terms of the quadruple considered in (2.6).

Definition 3.1. A *degenerating family of symmetric one-motives* over S is:

(a) an abelian scheme $p : \mathcal{A} \to S$ and a polarization $\lambda : \mathcal{A} \to \check{\mathcal{A}}$;

(b) a morphism $v : X \to \mathcal{A}(S)$, where X is a free $\mathbb{Z}$-module of finite rank; let $\check{v} = \lambda \circ v$;

(c) a symmetric trivialisation ψ of the inverse image by $(v, \check{v})$ of the Poincaré biextension of $\mathcal{A}_K$ and $\check{\mathcal{A}}_K$ by $\mathbb{G}_m$.

There is also a degeneracy condition for whose statement we refer to Brylinski (1983), 3.1.1.

From the data in (a) and (b), we can construct a semi-abelian variety $\mathcal{G}$ over S: let $\mathcal{T}$ be the constant split torus over S with $X^*(\mathcal{T}) = X$; then $\mathcal{G}$ is an extension of $\mathcal{A}$ by $\mathcal{T}$, such that, for all characters χ of T, $\chi_*(\mathcal{G})$ is an element of $\mathrm{Ext}^1(\mathcal{A}, \mathbb{G}_m)$ representing $\check{v}(\chi)$.

THEOREM 3.2. *There exists a semi-abelian scheme A over S, arising in a natural way from a degenerating one-motive, such that*

(a) the formal completion of A is the quotient of the formal completion of $\mathcal{G}$ by the group of periods $u(X)$;

(b) the restrictions to S_0 of the semi-abelian schemes A and $\mathcal{G}$ are canonically isomorphic.

PROOF: In the case that $\mathcal{G} = \mathcal{T}$ this was proved by Mumford (1972). Apparently, he also proved the general case, but never published it. There is a sketch of a proof in Brylinski (1983) and a detailed proof in Chai (1985).

Remark 3.3. In Faltings (1985) there is an important converse to (3.2).

Notes. The theorems in this section are due to Mumford (1972), Brylinski (1983), Faltings (1985), and Chai (1985). The most complete account is in Chai and Faltings (1989).

V. TOROIDAL COMPACTIFICATION

We explain the how to construct (smooth) toroidal compactifications of Shimura varieties, and suggest how the isomorphisms of Chapters II and III extend to these compactifications.

1. Torus embeddings. We review (without proofs) the construction in algebraic geometry on which the method of toroidal compactifications is based. Throughout this section, k will be an algebraically closed field, and "variety" will mean a reduced irreducible separated scheme locally of finite-type over k. All semigroups have zero elements and a subsemigroup of a (semi-) group contains the zero element of the (semi-) group.

Definitions. Let T be an d-dimensional torus over a field k. Write $M = X^*(T) \subset \Gamma(T, \mathcal{O}_T)$ and $N = X_*(T)$. For $r \in M$ let χ^r be the corresponding element of $\Gamma(T, \mathcal{O}_T)$, and for $a \in N$, let $\mu_a : \mathbb{G}_m \to T$ be the corresponding cocharacter. We have a pairing

$$\langle , \rangle : M \times N \to \mathbb{Z}, \quad \chi^r(\mu_a(t)) = t^{\langle r,a \rangle}.$$

As a k-algebra, $\Gamma(T, \mathcal{O}_T)$ is generated by $\{\chi^r \mid r \in M\}$. Moreover, if $r_1, \ldots, r_d$ is a basis for M, then

$$\Gamma(T, \mathcal{O}_T) = k[\chi_{r_1}, \chi_{r_1}^{-1}, \ldots, \chi_{r_d}, \chi_{r_d}^{-1}].$$

A *torus embedding* of T is an open immersion $T \hookrightarrow X$ of varieties together with an action of T on X whose restriction to T is the multiplication map. A morphism of torus embeddings is a homomorphism $f : X \to X'$ whose restriction to T is a homomorphism $T \to T'$ and which makes

$$\begin{tikzcd} T \times X \arrow[r] \arrow[d, "(F|T)\times f"'] & X \arrow[d, "f"] \\ T' \times X' \arrow[r] & X' \end{tikzcd}$$

commute. The torus embedding is said to be *affine* if X is affine.

Affine torus embeddings. Let $S \subset M$ be a finitely generated semigroup, and let $k[S]$ be the subalgebra of $\Gamma(T, \mathcal{O}_T)$ generated by $\{\chi^r \mid r \in S\}$. It is a finitely generated k-subalgebra of $\Gamma(T, \mathcal{O}_T)$, and its field of fractions is $k(T)$ if and only if S generates M (as a group). In this case, T acts on $X_S =_{\mathrm{df}} \operatorname{Spec} k[S]$, and $T \hookrightarrow X_S$ is an affine torus embedding. We have

$$X_S(k) = \operatorname{Hom}_*(S, k) =_{\mathrm{df}} \{x : S \to k \mid x(0) = 1, x(s+s') = x(s)x(s')\}.$$

Example 1.1. Let $T = \mathbb{G}_m^d$, so that $M = \mathbb{Z}^d$ and the coordinate ring of T, $k[T] = k[\chi_1, \chi_1^{-1}, \ldots]$. Let

$$S = \{(m_i, \ldots, m_d) \mid m_i \geq 0, i = 1, \ldots, s\}.$$

Then $\operatorname{Spec} k[S] = k^s \times (k^\times)^{d-s}$.

Let φ be a morphism $\mathbb{A}^1 - \{0\} \to X$; when φ extends to a morphism $\tilde{\varphi} : \mathbb{A}^1 \to X$, we say that $\lim_{t\to 0} \varphi(t)$ exists and equals $\tilde{\varphi}(a)$. With this definition, it is possible to describe X_S as the variety obtained from T by adding certain limit points: for each $a \in N$, $\lim_{t\to 0} \mu_a(t)$ exists in X_S if and only if $\langle a, S\rangle \geq 0$.

PROPOSITION 1.2. *(a) The map $S \mapsto (T \hookrightarrow X_S)$ defines a one-to-one correspondence between the set of finitely generated semigroups S in M generating M as a group and the set of isomorphism classes of affine torus embeddings of T.*

(b) An inclusion $S \subset S'$ defines a morphism $X_{S'} \hookrightarrow X_S$

(c) X_S is a normal variety if and only if S is a saturated in M, i.e., $m \in S$ whenever $rm \in S$ for some $r \in \mathbb{N}$, $r \neq 0$.

We want to patch affine torus embeddings together; for this it is convenient use different combinatorial data, so that the functor attaching a torus embedding to the data is covariant. A subset $\sigma \subset N_{\mathbb{R}}$ is called a *convex polyhedral cone* if there exist vectors $n_1, \ldots, n_s$ in $N_{\mathbb{R}}$ such that

$$\sigma = \{\sum_{i \geq 1} a_i n_i \mid a_i \in \mathbb{R},\, a_i \geq 0\}.$$

It is *rational* if the n_i can be chosen in N, and it is *strongly convex* if further $\sigma \cap (-\sigma) = 0$ (equivalently, σ contains no nonzero subspace of $N_{\mathbb{R}}$). The dimension of the subspace generated by σ is called the *dimension* of σ.

Let $\sigma = \sum \mathbb{R}_{\geq 0} n_i$ be a strongly convex rational polyhedral cone. If we remove redundant n_i's and require each to be primitive (that is, such that $rn_i \notin N$, $r \in \mathbb{Z}$, $r > 1$), then the set $\{n_1, \ldots, n_r\}$ is uniquely determined. These n_i are called the *fundamental generators* of σ.

The *dual* of σ is the convex rational polyhedral cone $\check{\sigma}$ in $M_{\mathbb{R}}$:

$$\check{\sigma} = \{r = M_{\mathbb{R}} \mid \langle r, a\rangle \geq 0, \text{ all } a \in \sigma\}.$$

PROPOSITION 1.3. *The map $\sigma \mapsto \check{\sigma} \cap M$ defines a one-to-one correspondence between the set of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ and the set of finitely generated semigroups $S \subset M$ generating M and saturated in M.*

For a convex rational polyhedral cone σ in $N_{\mathbb{R}}$, write X_σ for $\operatorname{Spec} k[\check{\sigma} \cap M]$. Note that for the cone $\sigma_0 = \{0\}$, $X_{\sigma_0} = T$. On combining the last two propositions, we obtain the following result.

COROLLARY 1.4. *The map $\sigma \mapsto X_\sigma$ defines a one-to-one correspondence between the set of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ and the set of affine normal torus embeddings of T.*

Remark 1.5. The following criterion allows us to reconstruct σ from X_σ: an element a of N lies in $\sigma \Leftrightarrow \lim_{t\to 0} \mu_a(t)$ exists in X_σ.

PROPOSITION 1.6. *The variety X_σ is nonsingular if and only if the fundamental generators of σ form part of a $\mathbb{Z}$-basis of N.*

A strongly convex rational polyhedral cone satisfying the condition in the proposition is said to be *nonsingular*.

The intersection of a strongly convex rational polyhedral cone σ with a hyperplane that does not meet the interior of σ is called a *face*, $\tau \prec \sigma$, of σ. There is then an r_0 in $\check{\sigma} \cap M$ such that

$$\tau = \{x \in \sigma \mid \langle r_0, x \rangle = 0\},$$

and τ is again a strongly convex rational polyhedral cone. The semigroup $\check{\tau} \cap M$ associated with τ is $\check{\sigma} \cap M + \mathbf{N}(-r_0)$.

PROPOSITION 1.7. *If τ and σ are strictly convex rational polyhedral cones and $\tau \subset \sigma$, then there is a natural morphism $X_\tau \to X_\sigma$ of torus embeddings; the morphism is an open immersion if and only if τ is a face of σ.*

On points, the map is the natural inclusion $\mathrm{Hom}_*(\check{\tau} \cap M, k) \hookrightarrow \mathrm{Hom}_*(\check{\sigma} \cap M, k)$ induced by $\check{\sigma} \cap M \hookrightarrow \check{\tau} \cap M$.

General torus embeddings. The last result suggests how to patch together X_σ for different σ.

Definition 1.8. A *fan* (formerly, rational partial polyhedral decomposition) of $N_{\mathbf{R}}$ is a nonempty collection $\Delta = \{\sigma\}$ of strongly convex rational polyhedral cones such that:

(i) every face of a cone in Δ is also in Δ;

(ii) if σ and σ' are in Δ, then $\sigma \cap \sigma'$ is a face of both σ and σ' .

The set $|\Delta| = \cup_{\sigma \in \Delta} \sigma$ is called the *support* of Δ, and Δ is said to be *complete* if $|\Delta| = N_{\mathbf{R}}$.

For example, the set of all faces of a strongly convex rational polyhedral cone is a fan. Let Δ be a fan in $N_{\mathbf{R}}$, and let

$$X_\Delta = \{(\sigma, \pi) \mid \sigma \in \Delta,\ \pi \in \mathrm{Hom}_*(\check{\sigma} \cap M, k)\}/\sim,$$

where $(\sigma, \pi) \sim (\sigma', \pi')$ if and only if π and π' are restrictions of a single element of $\mathrm{Hom}_*((\sigma \cap \sigma')^\vee \cap M, k)$.

PROPOSITION 1.9. *The space X_Δ has a unique structure of an algebraic variety for which the maps $X_\sigma \hookrightarrow X_\Delta$ are open immersions for all $\sigma \in \Delta$. In particular, $T = X_{\sigma_0} \hookrightarrow X_\Delta$ is an open immersion. There is a unique action of T on X_Δ extending its action on each X_σ.*

To summarize: we have attached to each fan in $N_{\mathbf{R}}$ a normal torus embedding $T \subset X_\Delta$.

Example 1.10. Let $N = \mathbf{Z}$, $\sigma = \mathbf{R}_{\geq 0} \subset N_{\mathbf{R}}$, $\Delta = \{\sigma, -\sigma, \{0\}\}$; then $X_\Delta = \mathbf{P}^1$.

THEOREM 1.11. *(a) X_Δ is of finite-type if and only if Δ is finite.*

(b) X_Δ is nonsingular if and only each X_σ is nonsingular.

(c) X_Δ is complete if and only if Δ is a finite and complete fan.

(d) X_Δ is quasi-projective if and only if Δ is finite and there is a continuous real-valued convex function on the convex hull of $|\Delta|$ such that

(i) $f|\sigma$ is $\mathbb{R}$-linear, all $\sigma \in \Delta$;

(ii) f takes integer values on $N \cap |\Delta|$;

(iii) for each $\sigma \in \Delta$, there is an $r_\sigma \in M$ and an $n_\sigma > 0$ such that $n_\sigma f \geq r_\sigma$ on $|\Delta|$ and

$$\sigma = \{a \in N_{\mathbb{R}} \mid \langle r_\sigma, a\rangle = n_\sigma f(a)\}.$$

The function f in (iii) is called a *polar function.* It defines a T-equivariant ample invertible sheaf on X_Δ.

Remark 1.12. (a) The X_σ for $\sigma \in \Delta$ are the T-stable affine open subsets of X_Δ. In particular, X_Δ is affine if and only if there is a $\sigma \in \Delta$ such that Δ coincides with the set of faces of π.

(b) The description given above for the k-points of X_Δ extends to a description of the functor of k-schemes defined by X_Δ (see Ash et al. 1975, p10, except note that they forget to pass to the equivalence classes).

PROPOSITION 1.13. *Each torus embedding $T \subset X$ with X normal is isomorphic to the torus embedding defined by a fan Δ in $X_*(T) \otimes \mathbb{R}$, and Δ is uniquely determined.*

Equivariant maps. A *map of fans* $\varphi : (N', \Delta') \to (N, \Delta)$ is a homomorphism $\varphi : N' \to N$ such that the image under $\varphi_{\mathbb{R}}$ of each $\sigma' \in \Delta'$ is contained in a $\sigma \in \Delta$.

PROPOSITION 1.14. *Let $\varphi : (N', \Delta') \to (N, \Delta)$ be a map of fans; the map $T_{N'} \to T_N$ defined by φ extends uniquely to a morphism $\varphi_* : X_{\Delta'} \to X_\Delta$, and ϕ_* is equivariant. Each morphism of torus embeddings $X_{\Delta'} \to X_\Delta$ arises in this way from a unique map of fans.*

PROPOSITION 1.15. *The morphism φ_* is proper and birational if and only if $\varphi : N' \to N$ is an isomorphism and Δ' is a locally finite subdivision of Δ.*

Rationality of torus embeddings over subfields. Let $\tau : k \hookrightarrow k'$ be an inclusion of k into a second algebraically closed field k'. Then τ defines an isomorphism $X_*(T) \to X_*(\tau T)$, and a fan Δ in $X_*(T) \otimes \mathbb{R}$ is mapped to a fan $\tau\Delta$ in $X_*(\tau T) \otimes \mathbb{R}$. Clearly, $\tau(X_\Delta) = X_{\tau\Delta}$ as torus embeddings of τT.

Now suppose that T is defined over a subfield k_0 of k over which k is Galois. Then $\mathrm{Gal}(k/k_0)$ acts on N (through its action on T), and descent theory shows that a quasi-projective normal torus embedding $T \hookrightarrow X_\Delta$ is defined over k_0 if and only if Δ is stable under the action of $\mathrm{Gal}(k/k_0)$ on $N_{\mathbb{R}}$.

Toroidal embeddings. Let Y be a normal variety, and let U be a smooth open subset of Y. We say that $U \subset Y$ is a *toroidal embedding* if it is a torus embedding locally for the étale topology. We mean by this that for every closed point y of Y there is an open neighbourhood Y' of y, a normal affine torus embedding $T \subset X$, and an étale map $\pi : Y' \to X$ such that $\pi^{-1}(T) = U \cap Y'$:

$$\begin{array}{ccccc} Y & \overset{\text{open}}{\hookleftarrow} & Y' & \xrightarrow{\text{étale}} & X \\ \cup & & \cup & & \cup \\ U & \hookleftarrow & U \cap Y' & \longrightarrow & T. \end{array}$$

Compactification of torsors. Let V be a variety, and let P be a T-torsor over V. For any torus embedding $T \hookrightarrow X$ we can define:

$$P \times {}^T X = (P \times X)/\sim, \ (pt, x) \sim (p, tx), \ p \in P, \ x \in X, \ t \in T.$$

This is a variety over X. The choice of a point p in the fibre P_v of P over a closed point $v \in V$ defines an isomorphism

$$\begin{array}{ccc} T & \hookrightarrow & X \\ \downarrow\approx & & \downarrow\approx \\ P_v & \hookrightarrow & (P \times {}^T X)_v. \end{array}$$

A similar construction can be made when V is a complex manifold. In this case, $P \times {}^T X$ is a fibre bundle over V with standard fibre X (see Kobayashi and Nomizu, 1963).

Notes. Detailed proofs of the results in this section can be found in Kempf et al. (1972) and Oda (1978), (1987).

2. Study of the boundary of symmetric Hermitian domains. There is a very elaborate theory concerning the boundaries of Hermitian symmetric domains. We can include only a very brief sketch.

Rational boundary components. Let D be a symmetric Hermitian domain. Since we are interested in its boundary, we assume D to be noncompact. There then exists a semisimple group G over $\mathbb{Q}$ such that $G(\mathbb{R})^{+} = \mathrm{Aut}(D)^{+}$.

As was explained in (III.1), there is a canonical embedding $\beta : D \hookrightarrow \check{D}$ of D into its compact dual. The closure $\bar{D}$ of D in $\check{D}$ is called the *natural compactification* of D. The action of $G(\mathbb{R})^{+}$ on D extends to a continuous action on $\bar{D}$. The space $\bar{D}$ can be decomposed according to the equivalence relation generated by the following relation: $x \sim y$ if there is a holomorphic map $\lambda : D_1 \to \check{D}$ from the unit disk D_1 into $\check{D}$ such that $\{x, y\} \subset \lambda(D_1) \subset \bar{D}$. The equivalence classes are called the *boundary components* of D. Note that this definition allows D itself to be an boundary component of $\bar{D}$ (called the *improper* boundary component).

The *normalizer* of a boundary component F is the subgroup $\mathcal{N}$ of $G(\mathbb{R})^{+}$ containing those g such that $gF = F$. The component F is said to be *rational* if there is a subgroup $N^{F} \subset G$ (defined over $\mathbb{Q}$ such that $\mathcal{N}^{+} = N^{F}(\mathbb{R})^{+}$.

PROPOSITION 2.1. *(a) When G is simple, the map $F \mapsto N^{F}$ is a bijection from the set of proper rational boundary components of D to the set of maximal parabolic subgroups of G.*

(b) Suppose $G = G_1 \times \cdots \times G_m$ with each G_i simple, and let $D = D_1 \times \cdots \times D_m$ be the corresponding decomposition of X. The rational boundary components F of D are products $F_1 \times \cdots \times F_m$ with each F_i a rational boundary component of D_i, and the normalizer of such an F is the product of the normalizers of the F_i

PROOF: See Baily and Borel (1966), 3.7.

From now on, we assume G to be simple (over $\mathbb{Q}$).

Example 2.2. Let (V, ψ) be a symplectic space, let $G = \mathrm{Sp}(V, \psi)$, and let D be the corresponding Siegel upper-half-space. For any totally isotropic subspace W of V, the stabilizer N of W in V is a maximal parabolic subgroup of G, and all such subgroups are of this form. The boundary component F corresponding to N is isomorphic to the Siegel upper-half-space defined by the symplectic space $(W^{\perp}/W, \bar{\psi})$.

For example, if $\dim V = 2$, then the totally isotropic subspaces are the (rational) lines in V. They are in one-to-one correspondence with the points of $\mathbb{P}^1(\mathbb{Q})$. When D is realized as the open unit disk, then the rational boundary components are the points on the circle that lie on a line through the origin with rational slope.

Cayley filtrations. For each point $x \in D$, there is homomorphism $h_x : \mathbb{S} \to G_{\mathbb{R}}$ such that $h_x(z)$ fixes x and acts on $\mathrm{Tgt}_x(D)$ as multiplication by z^2. The map $x \mapsto h_x$ identifies D with a $G(\mathbb{R})^+$-conjugacy class of maps. For a representation (V, ξ) of $G_{\mathbb{R}}$, $\xi \circ h_x$ defines a Hodge structure on V and a (decreasing) Hodge filtration $F_x^{\cdot}$ on $V(\mathbb{C})$.

Definition 2.3. A filtration $W.$ of $\mathbf{Rep}_{\mathbb{Q}}(G)$ is said to be *Cayley* if for all $x \in D$ and all representations $\xi : G \to GL(V)$, the filtrations $W.$ and $F_x^{\cdot}$ of V define a mixed Hodge structure on V.

PROPOSITION 2.4. *If $W.$ is a Cayley filtration, then W_0G is a maximal parabolic subgroup of G, and every maximal parabolic subgroup of G is associated in this way with a unique Cayley filtration.*

PROOF: See Deligne (1973), 3.1.13.

Thus each rational boundary component F defines a Cayley filtration $W.$ of $\mathbf{Rep}_{\mathbb{Q}}(G)$. Deligne (ibid. 3.1.14) shows that for each F, there is a unique cocharacter w_F of G splitting the corresponding Cayley filtration and such that $(\operatorname{ad} h(i)) \circ w_F = w_F^{-1}$.

THEOREM 2.5. *Fix a base point $o \in D$ and a rational boundary component F of $\bar{D}$. Then there exists a unique homomorphism*

$$\varphi_F : U^1 \times SL(2, \mathbb{R}) \to G(\mathbb{R})$$

such that

(i) $\varphi_F(e^{i\theta}, r(\theta)) = h_o(e^{i\theta})$, $r(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$,

(ii) $\varphi_F(1, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}) = w_F(\lambda)$, $\lambda \in U^1$.

PROOF: Deligne (1973), 3.1.14.

Remark 2.6. (a) Let H be the upper-half-plane. There is a holomorphic map $f_F : H \to D$ that is equivariant for φ_F and such that $f_F(i) = o$ and $f_F(\infty) \in F$ (Ash et al. 1975, p199).

(b) Since G is simple it can be written $G = \operatorname{Res}_{F/\mathbb{Q}} G'$ with G' an absolutely simple group over a totally real field F. Choose a point $o \in D$ such that h_o factors through $T(\mathbb{R})$ with T a maximal torus in G. If E is a CM-field splitting T', then φ_F is defined over the maximal totally real subfield of E (because both h_0 and w_F are).

The structure of N^F. Fix a base point $o \in D$ and a rational boundary component F. The Hodge structure on $\mathfrak{g}$ defined by h_0 is of type $\{(-1,1),(0,0),(1,-1)\}$ (cf. II.1). It follows that the nonzero Hodge numbers h^{pq} of the mixed Hodge structure $(\mathfrak{g}, W_., F_o^{\cdot})$ satisfy $|p|$, $|q| \leq 1$. The action of w_F therefore defines a grading:

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{1} \oplus \mathfrak{g}^{2}.$$

There are attached to F the following algebraic groups over $\mathbb{Q}$:

$N^F = W_0 G$; $\operatorname{Lie} N^F = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0}$.

$W^F = W_{-1}G =$ unipotent radical of N^F; $\operatorname{Lie} W^F = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1}$.

$U^F = W_{-2}G =$ centre of W^F; this is an abelian group, which we can identify with its Lie algebra $\mathfrak{g}^{-2}$.

$Z(w_F) =$ centralizer of w_F in N^F; $\operatorname{Lie}(w_F) = \mathfrak{g}^0$, and $N^F = W^F \rtimes Z(w_F)$.

$V^F = W^F/U^F$; this is an abelian group, which we can identify with its Lie algebra $\mathfrak{g}^{-1}$. Write $\mathfrak{g}_\ell = [\mathfrak{g}^2, \mathfrak{g}^{-2}]$, and $\mathfrak{g}_h =$ orthogonal complement $[\mathfrak{g}^2, \mathfrak{g}^{-2}]$ in $\mathfrak{g}^0$. The decomposition $\mathfrak{g}^0 = \mathfrak{g}_\ell + \mathfrak{g}_h$ can be integrated to an isogeny $G_h \times G_\ell \to Z(w_F)$. In summary:

$$\begin{array}{rl}
W^F \rtimes Z(w_F) = & N^F \\
 & \mid \sim G_h \times G_\ell \\
 & W^F \\
 & \mid \quad V^F \\
 & U^F \\
 & \mid \\
 & \{1\}
\end{array}$$

PROPOSITION 2.7. *(a) F is a symmetric Hermitian domain; G_h is semisimple, and*

$$G_h(\mathbb{R})^+/(\text{maximal compact subgroup}) = \mathrm{Aut}(F)^+.$$

(b) The morphism φ_W sends U^1 into G_h, and it sends $SL_2(\mathbb{R})$ into G_ℓ; moreover, $\varphi_W|U^1 : U^1 \to G_h(\mathbb{R})$ defines the complex structure on F.

(c) G_ℓ is reductive without compact factors.

(d) The centralizer of F, $\mathcal{Z} = \{g \in G(\mathbb{R}) \mid gx = x \text{ all } x \in F\}$, has identity component $G_\ell \times W^F$.

(e) $G_h \cdot W^F$ centralizes U^F.

PROOF: Ash et al. (1975), III.3.

Example 2.8. With the notations of (2.2), $G_h = \mathrm{Sp}(W^{\perp}/W, \bar{\psi})$, $G_\ell = GL(W)$, and $M^F = 0$. Moreover, U^F is the space of symmetric bilinear forms on $V(\mathbb{R})$.

The canonical self-dual open cone in $U^F(\mathbb{R})$. In addition to the closed cones of §1, we shall need to consider *open cones* in real vector spaces. Such a cone C in a real space V is said to be *self-dual* if there exists a positive-definite inner form $\langle\, , \rangle$ on V with the property that $x \in C$ if and only if $\langle x, y\rangle > 0$ whenever $0 \neq y \in \bar{C}$ (closure of C). The cone is said to be *homogeneous* if the group $\mathrm{Aut}(V, C)$ of automorphisms of V stabilizing C acts transitively on C.

Example 2.9. Every homogeneous self-adjoint cone can be written as a product of indecomposable cones. Apart from one family of semi-classical cones and one exceptional cone, every indecomposable homogeneous self-adjoint cone is isomorphic to a cone in the following list:

(i) the cone of positive-definite real symmetric matrices;
(ii) the cone of positive-definite Hermitian complex matrices;
(iii) the cone of positive-definite Hermitian quaternion matrices.

The Killing form B defines a Hermitian form on $\mathfrak{g}_{\mathbb{C}}$,

$$B'(x, y) = -B(x, \iota y), \quad x, y \in \mathfrak{g}_{\mathbb{C}},$$

which restricts to a positive-definite form on $\mathfrak{u}^F$. The isomorphism $\exp : \mathfrak{u}^F \to U^F$ allows us to transfer this to U^F.

Define Ω_F to be the point $\varphi_F(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$ of U^F. Then the orbit of Ω_F in $U^F(\mathbb{R})$ under $G_\ell(\mathbb{R})$,

$$C(F) = \{g\Omega_F g^{-1} \mid g \in G_\ell(\mathbb{R})\},$$

is a homogeneous open cone in $U_F(\mathbb{R})$, which is self-dual relative to B'.

Example 2.10. In the situation of (2.2), $C(F)$ is the cone of all positive-definite bilinear forms on W.

Definition of Siegel domains.

Definition 2.11. Let U be a real vector space and let C be an open convex cone in U whose closure does not contain an entire straight line. Then

$$S = \{z \in U(\mathbb{C}) \mid \mathrm{Im}(z) \in C\} = U + \mathrm{i}C$$

is a *tube domain* (or Siegel domain of the first kind).

Let U be a real vector space and V a complex vector space; a real-bilinear map $V \times V \to U(\mathbb{C})$ is said to be *semi-Hermitian* if it can be written as the sum of a symmetric complex-bilinear map and a Hermitian map.

Definition 2.12. Let U be a real vector space, let V be a complex vector space, and let $\mathcal{D}$ be a bounded domain in some space $\mathbb{C}^k$; let $C \subset U$ be a cone satisfying the conditions of (2.11). Suppose that for each $t \in D$ there is given a nondegenerate semi-Hermitian form L_t on V with values in U. Then

$$S = \{w = (z, v, t) \in U \times V \times \mathcal{D} \mid \operatorname{Im}(z) - \operatorname{Re}(L_t(v, v)) \in C\}$$

is a *Siegel domain (of the third kind).* Thus a Siegel domain can be thought of as a family of tube domains parametrized by $V \times \mathcal{D}$.

Realization of D as a Siegel domain. We now describe the realization of D as a Siegel domain of the third kind, attached to the component F. Let

$$D(F) = U^F(\mathbb{C}) \cdot D = \cup_{g \in U^F(\mathbb{C})} gD \subset \check{D}.$$

Example 2.13. In the situation of 2.2, $D(F)$ is the set of maximal isotropic subspaces $F^0 \subset V$ such that $(V_., W_., F^{\cdot})$ is a mixed Hodge structure and $\bar{\psi}$ is a polarization of $W/W^{\perp}$. Here $W_.$ and $F^{\cdot}$ are the filtrations:

$$0 \subset W \subset W^{\perp} \subset V, \quad V = F^{-1}V \supset F^0 \supset F^1 V = 0.$$

There is a $N^F(\mathbb{R}) \cdot U^F(\mathbb{C})$-equivariant map $\Phi_F : D(F) \to U^F(\mathbb{C})$ such that $\mathcal{D} = \Phi_F^{-1}(C)$. The space $D(F)$ can be decomposed by means of two successive fibrations:

$$\begin{array}{c|c}
 & D(F) \\
 & \downarrow \pi_1 \\
\pi_F & D(F)' = U^F(\mathbb{C}) \backslash D(F) \\
\downarrow & \downarrow \pi_2 \\
 & F
\end{array}$$

Moreover,

$D(F)$ is a fibre bundle over $D(F)'$ for the complex group $U^F(\mathbb{C})$;
$D(F)' \to F$ is a principal C^∞-fibration for the group $V^F(\mathbb{R})$.
Both fibrations can be trivialized,

$$D(F) \approx U^F(\mathbb{C}) \times D(F)' \approx U^F(\mathbb{C}) \times V^F(\mathbb{C}) \times F,$$

and with the choice of such a decomposition, Φ_F can be expressed

$$\Phi_F(z, v, t) = \mathrm{Im}(z) - h_t(v, v), \; z \in U^F(\mathbb{C}), \; v \in V^F(\mathbb{C}), \; z \in F$$

with h_t a real bilinear form $V^F(\mathbb{R}) \times V^F(\mathbb{R}) \to U^F$ depending real-analytically on t. Thus D is equal to

$$\{(z, v, t) \mid z \in U^F(\mathbb{C}), \; v \in V^F(\mathbb{C}), \; t \in F, \mathrm{Im}(z) - h_t(v, v) \in C(F)\},$$

which realizes it as a Siegel domain.

Algebraicity of the quotient of $D(F)$ by a discrete group. An arithmetic subgroup Γ of $G(\mathbb{R})$ is said to be *neat* if it consists of elements g such that, for one (hence every) faithful complex representation ξ of G, the subgroup of $\mathbb{C}^\times$ generated by the eigenvalues of $\xi(g)$ is torsion-free. In particular, a neat subgroup is torsion-free.

Choose a neat arithmetic subgroup Γ of $G(\mathbb{R})$ (every arithmetic subgroup contains a subgroup of finite index that is neat), and define:
$\Gamma(F) = \Gamma \cap \mathcal{N}$; it is a discrete subgroup of N^F;
$\Gamma'(F)$ = subgroup of $\Gamma(F)$ of elements acting trivially (by conjugation) on U^F;
$\Gamma_h(F)$ = image of $\Gamma(F)$ in $G_h(\mathbb{Q})$; it is a neat subgroup of $G_h(\mathbb{Q})$, and so $\Gamma_h(F)\backslash F$ is a locally symmetric variety.
The quotient $U^F(\mathbb{C})/(U^F(\mathbb{C})\cap\Gamma)$ is compact, and $U^F(\mathbb{C})\cap\Gamma$ is discrete in $U^F(\mathbb{C})$; therefore $U^F(\mathbb{C}) \cap \Gamma$ is a lattice in $U^F(\mathbb{C})$, and $T^F = U^F(\mathbb{C})/(U^F(\mathbb{C}) \cap \Gamma)$ is a complex torus.

THEOREM 2.14. *The quotient $\Gamma'(F)\backslash D(F)$ has a canonical structure of an algebraic variety for which the map $\Gamma'(F)\backslash D(F) \to \Gamma_h(F)\backslash F$ is a morphism of algebraic varieties. In fact, $\Gamma'^F\backslash D^F$ is a torus bundle (with fibres $T^F(\mathbb{C})$) over an abelian scheme over $\Gamma_h(F)\backslash F$.*

PROOF: This is proved in Brylinski (1979). (See also Brylinski (1983), 2.3.2.5, and Chapter VI below.)

Remark 2.15. The algebraic structure in (2.14) is canonical, but it is not unique: there is no analogue of the Borel extension theorem (cf. II.1.1).

Example: the Siegel case. We return to the situation of (2.2). Choose a lattice $V(\mathbb{Z})$ in V such that ψ takes integral values and has discriminant one on $V(\mathbb{Z})$, and let $\mathrm{Sp}(\mathbb{Z})$ be the subgroup of $\mathrm{Sp}(V, \psi)$ preserving $V(\mathbb{Z})$.The quotient $\mathrm{Sp}(\mathbb{Z})\backslash D$ is the moduli variety for polarized abelian varieties in the principal series. Fix an isotropic subspace W of V, and define the filtration $W.$ as in (2.13). The form ψ induces on $\mathrm{Gr}_{-1}(V(\mathbb{Z}))$ a skew-symmetric form $\bar{\psi}$ of discriminant 1. Set $\dim \mathrm{Gr}_{-1} V = 2g_0$. We have:

(a) F is the space of Hodge structures of type $\{(-1,0),(0,-1)\}$ on $\mathrm{Gr}_{-1}(V)$ for which $\bar{\psi}$ is a polarization.

(b) $D(F)$ is the space of maximal isotropic subspaces F^0 of $V(\mathbb{C})$ such that $(V, W., F^{\cdot})$ is a mixed Hodge structure and ψ is a polarization of $\mathrm{Gr}_{-1}(V)$.

Let Γ' be the subgroup of $\mathrm{Sp}(\mathbb{Z})$ of elements that respect the filtration and act trivially on $\mathrm{Gr}_0(V)$.

(c) The quotient $\Gamma'\backslash D(F)$ is the (coarse) moduli variety for symmetric one-motives $(A, \lambda, X, v, \delta)$ with (A, λ) a principally polarized abelian variety of dimension $2g_0$, X the abelian group $\mathrm{Gr}_0(V(\mathbb{Z}))$, v a homomorphism $X \to A(\mathbb{C})$, and δ a symmetric trivialization of the Poincaré biextension (see IV.2.6).

(d) The quotient $\Gamma'\backslash D(F)'$ is the (coarse) moduli variety for the quadruples (A, λ, X, v).

(e) The quotient $\Gamma'\backslash F$ is the (coarse) moduli variety for principally polarized abelian varieties of dimension g_0.

The maps

$$\Gamma'\backslash D(F) \to \Gamma'\backslash D'(F) \to \Gamma'\backslash F$$

correspond to

$$(A, \lambda, X, v, \delta) \mapsto (A, \lambda, X, v) \mapsto (A, \lambda).$$

Notes. Piatetski-Shapiro (1966) showed how to realize all the classical symmetric Hermitian domains as Siegel domains of the third kind. Wolf and Korányi (1965) gave a more uniform treatment that includes the nonclassical domains. There are expositions of (parts of) the material in this section in Baily and Borel (1966), Deligne (1973), Ash et al. (1975), Satake (1980), and Brylinski (1983).

3. Toroidal compactification of locally symmetric varieties. The results of the last two sections, allow us to construct toroidal compactifications of locally symmetric varieties.

We use the same notations as in §2 (except that we no longer require G to be $\mathbb{Q}$-simple). Thus D is a symmetric Hermitian domain, G is an algebraic group over $\mathbb{Q}$ with $G(\mathbb{R})^+ = \mathrm{Aut}(D)^+$, F is a rational boundary component of D, and N^F, W^F, and U^F are certain subgroups of G attached to F. Recall that we have a canonical self-adjoint open cone $C(F)$ in $U^F(\mathbb{R})$. We choose a neat arithmetic subgroup Γ of $G(\mathbb{R})^+$, and define $\bar{\Gamma}(F)$ to be the image of $\Gamma(F)$ in $\mathrm{Aut}(U^F)$. As in §2, T^F is the torus over $\mathbb{C}$ with $X_*(T) = U^F(\mathbb{Z}) =_{\mathrm{df}} U^F(\mathbb{C}) \cap \Gamma$. Finally, we write S for the locally symmetric variety $\Gamma\backslash D$.

Definition 3.1. A fan Δ in $U^F(\mathbb{R})$ is said to be $\bar{\Gamma}(F)$-admissible if
(a) $\gamma \in \bar{\Gamma}(F)$, $\sigma \in \Delta \Rightarrow \gamma\sigma \in \Delta$;
(b) the number of classes of cones mod $\bar{\Gamma}(F)$ is finite;
(c) $C(F) \subset |\Delta| \subset C(F)^-$ (closure of $C(F)$).

Note that $X_*(T) \otimes \mathbb{R} = U^F(\mathbb{R})$. Therefore a $\bar{\Gamma}(F)$-admissible fan gives a torus embedding $T^F \subset X^F_\Delta$. As $U^F(\mathbb{Z})\backslash D(F)$ is a principal bundle for T^F over $D(F)'$, we can construct a partial compactification,

$$(U^F(\mathbb{Z})\backslash D(F))_\Delta = (U^F(\mathbb{Z})\backslash D(F)) \times^{T^F} X^F_\Delta,$$

as at the end of §1. This is a fibre bundle over $D(F)'$ with fibres X^F_Δ. Define $(U^F(\mathbb{Z})\backslash D)_\Delta$ to be the interior of the closure of $U^F(\mathbb{Z})\backslash D$ in $(U^F(\mathbb{Z})\backslash D(F))_\Delta$. Because Δ is invariant under $\bar{\Gamma}(F)$, $\Gamma(F)$ acts on $(U^F(\mathbb{Z})\backslash D(F))_\Delta$, and it can be shown that $\Gamma(F)$ acts properly discontinuously on $(U^F(\mathbb{Z})\backslash D)_\Delta$.

Definition 3.2. A family of fans $\Delta = (\Delta^F)$, F running over the rational boundary components of D, is Γ-*admissible* if
(a) each Δ^F is $\bar{\Gamma}(F)$-admissible;
(b) for $\gamma \in \Gamma$, $\gamma\Delta^F = \Delta^{\gamma F}$ (note that γ defines an isomorphism $\gamma : C(F) \to C(\gamma F)$);
(c) if $F \supset F'$, $\Delta^{F'} = \{\sigma \cap C(F') \mid \sigma \in \Delta\}$ (note that $C(F')^- = C(F)^- \cap U(F')$).

THEOREM 3.3. *For every Γ-admissible family of fans $\Delta = (\Delta^F)$, there is a unique normal separated complex analytic variety $(\Gamma\backslash D)_\Delta$ containing $\Gamma\backslash D$ as an open dense set and such that:*

(a) for every rational boundary component F of D, there is an open analytic morphism π_F making the following diagram commute:

$$\begin{array}{ccc} U^F(\mathbb{Z})\backslash D & \hookrightarrow & (U^F(\mathbb{Z})\backslash D)_{\Delta^F} \\ \Big| & & \Big|\pi_F \\ \Gamma\backslash D & \hookrightarrow & (\Gamma\backslash D)_\Delta; \end{array}$$

(b) $(\Gamma\backslash D)_\Delta = \cup Im(\pi_F)$. *Moreover,* $(\Gamma\backslash D)_\Delta$ *has a unique structure of a complete algebraic space compatible with its analytic structure, and there is a natural morphism* $(\Gamma\backslash D)_\Delta \to (\Gamma\backslash D)^-$ *that restricts to the identity map on* $\Gamma\backslash D$.

PROOF: This is the main theorem of Ash et al. (1975) (ibid. p253).

The algebraic space $(\Gamma\backslash D)_\Delta$ in the theorem is called the *toroidal compactification* of $\Gamma\backslash D$ defined by Δ.

An algebraic space is the quotient of a scheme by an étale equivalence relation (see Knutson (1971) for a full account of the theory of algebraic spaces). In this article, the distinction between a scheme and an algebraic space will not be important, and we shall ignore it. The next two results show that Δ can be chosen so that the toroidal compactification is in fact a projective variety.

Let $U = \cup U^F$ and $C = \cup C(F)$ (unions over the rational boundary components of D).

Definition 3.4. Let $\Delta = (\Delta^F)_F$ be a Γ-admissible family of fans.

(a) Δ is *nonsingular* if every cone in every Δ_F is nonsingular (see 1.6);

(b) Δ is *projective* if there exists a Γ-invariant continuous convex piecewise linear function $f : C \to \mathbb{R}$ such that $f|U^F$ is a polar function for each F (see 1.11).

THEOREM 3.5. *(a) If* Δ *is nonsingular, then* $(\Gamma\backslash D)_\Delta$ *is nonsingular.*

(b) If Δ *is projective, then*$(\Gamma\backslash D)_\Delta \to (\Gamma\backslash D)^-$ *is the normalization of the blowing up of* $(\Gamma\backslash D)$ *along a sheaf of ideals* $\mathcal{I}$ *such that* $\mathcal{O}/\mathcal{I}$ *has support on* $(\Gamma\backslash D)^- - \Gamma\backslash D$. *In particular,* $(\Gamma\backslash D)_\Delta$ *is projective.*

PROOF: The first statement follows from (1.11). The second is a theorem of Tai (see Ash et al. (1975), IV.2.1).

PROPOSITION 3.6. *(a) There exist projective* Γ*-admissible families of fans.*

(b) Every Γ*-admissible fan has a refinement that is nonsingular.*

PROOF: (a) See Ash et al. (1975), p310.

(b) In Kempf et al. (1972), p32, this is proved for torus embeddings of finite type, but essentially the same proof works in the present context.

One can show, more precisely, that every toroidal compactification is dominated by a nonsingular toroidal compactification whose boundary is a divisor with normal crossings—we shall refer to such a compactification as a *smooth toroidal compactification.*

Remark 3.7. (a) The sheaf of ideals $\mathcal{I}$ in (3.5b) has a precise description in terms of the function f (see Ash et. al. 1975, p312).

(b) In Ash et. al (1975), p287, there is a more intrinsic statement of the main theorem.

Notes. Toroidal compactification were introduced independently by Mumford and Satake (see Mumford 1975 and Satake 1973). The theory was worked out in detail by Mumford and his collaborators, Ash, Kempf, Knudsen, Rapoport, Saint-Donat, and Tai, in (Kempf et al. 1972) and (Ash et al. 1975).

4. Toroidal compactification of Shimura varieties. We extend the results of the last section to Shimura varieties.

Toroidal compactification of connected Shimura varieties. Let (G, X^+) be a pair satisfying the axioms (II.1.3). The group G^{ad} plays the role of G in the previous section. Let Γ be a neat arithmetic subgroup of $G^{\mathrm{ad}}(\mathbb{Q})^+$ containing the image of a congruence subgroup of $G(\mathbb{Q})^+$. A Γ-admissible fan Δ will also be Γ'-admissible for any arithmetic subgroup $\Gamma' \subset \Gamma$, and the morphism $\Gamma'\backslash X^+ \to \Gamma\backslash X^+$ extends to a morphism $(\Gamma'\backslash X^+)_\Delta \to (\Gamma\backslash X^+)_\Delta$. We write $\mathrm{Sh}^0(G,X)_\Delta$ for the projective system $(\Gamma\backslash X^+)_\Delta$, where Γ runs over the neat arithmetic subgroups containing the image of a congruence subgroup.

Unfortunately, the action of $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$ on $\mathrm{Sh}^0(G,X)$ does not extend to $\mathrm{Sh}^0(G,X)_\Delta$. However, we have the following observation of Faltings and Stuhler.

LEMMA 4.1. *Let Γ and Γ' be neat arithmetic subgroups of $G^{\mathrm{ad}}(\mathbb{Q})^+$ containing the image of a congruence subgroup, and let $\gamma_1, \ldots, \gamma_n \in G(\mathbb{Q})^+$ be such that $\gamma_i^{-1}\Gamma\gamma_i \subset \Gamma'$; then for any pair of smooth toroidal compactifications $(\Gamma\backslash X^+)_\Delta$ and $(\Gamma'\backslash X^+)_{\Delta'}$ of $\Gamma\backslash X^+$ and $\Gamma'\backslash X^+$, there exists a smooth compactification $(\Gamma\backslash X^+)_{\Delta''}$ of $\Gamma\backslash X^+$ and maps:*

$$(\Gamma\backslash X^+)_{\Delta''} \to (\Gamma\backslash X^+)_\Delta \quad \textit{restricting to id on } \Gamma\backslash X^+\textit{, and}$$
$$(\Gamma\backslash X^+)_{\Delta''} \to (\Gamma'\backslash X^+{}_{\Delta'} \quad \textit{restricting to } \gamma_i \textit{ on } \Gamma\backslash X^+.$$

PROOF: Stated in Faltings (1984).

Thus, if we define $\mathrm{Sh}^0(G,X)^*$ to be the projective system $(\Gamma\backslash X^+)_\Delta$, where Γ runs over the neat arithmetic subgroups of $G^{\mathrm{ad}}(\mathbb{Q})^+$ containing the image of a congruence subgroup of $G(\mathbb{Q})^+$ and (for each Γ) Δ runs over the Γ-admissible families of fans for which $(\Gamma\backslash X^+)_\Delta$ is a smooth toroidal compactification, then the action of $G^{\mathrm{ad}}(\mathbb{Q})^+$ on $\mathrm{Sh}^0(G,X)$ extends to $\mathrm{Sh}^0(G,X)^*$. By continuity, we obtain an action of $G^{\mathrm{ad}}(\mathbb{Q})^+$ on $\mathrm{Sh}^0(G,X)^*$.

Toroidal compactification of Shimura varieties. Let (G,X) be a pair defining a Shimura variety, and assume that the weight w_X is defined over $\mathbb{Q}$ (this is true for all naturally occurring Shimura varieties with boundary). Choose a connected component X^+ of X. Corresponding to a boundary component F of X^+, we obtain a Cayley filtration w^F of G^{ad}. It follows from results in Deligne (1973) that w^F lifts to a filtration w of $G_{\mathbb{C}}$, and that w can be normalized so that $(w_X \cdot w^{-1})(\mathbb{G}_m) \subset G^{\mathrm{der}}$ (i.e., w and w_X become equal when composed with $G_{\mathbb{C}} \to (G/G^{\mathrm{der}})_{\mathbb{C}}$). Because the map $G \to G^{\mathrm{ad}} \times (G/G^{\mathrm{der}})$ has finite kernel, w is uniquely determined, and because w^F and w_X are defined over $\mathbb{Q}$, so also is w. Moreover, for any representation (V,ξ) of G, the filtrations defined by w and $F_x^{\cdot}$ form a mixed Hodge structure on V (according to (IV.1.3), this has to be checked only for representations factoring through $G^{\mathrm{ad}} \times (G/G^{\mathrm{der}})$, and for these it is obvious). These remarks suggest the following definition.

Definition 4.2. A Cayley filtration $W.$ on G is *admissible* if the filtration on G/G^{der} is that defined by w_X.

Now fix an admissible Cayley filtration $W.^F$ of G. Here the F is simply an index. Set

$$N^F = W_0^F(G),\ W^F = W_{-1}^F(G),\ U^F = W_{-2}^F(G).$$

Note that $Z(G) \subset Z(w)$ for any w splitting $W.^F$, and so $Z(G) \cap W^F = \{1\}$. Therefore W^F and U^F are mapped isomorphically onto their images in G^{ad}.

Choose a connected component X^+ of X, and let K be a compact open subgroup of $G(\mathbb{A}_f)$. For any $g \in G(\mathbb{A}_f)$, let Γ_g be the image in $G^{\mathrm{ad}}(\mathbb{Q})^+$ of the group $gKg^{-1} \cap G(\mathbb{Q})_+$. As in §3, associated with Γ_g we have groups $\Gamma_g(F)$, $\Gamma'_g(F)$ and $\bar{\Gamma}_g(F)$, and we have a canonical cone $C(F) \subset U^F(\mathbb{C})$. Let $\mathcal{C}$ be a set of representatives for the finite set $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$ (see II.2).

Definition 4.3. A fan $\Delta \subset C(F)$ is said to be $\bar{\Gamma}(F)$-admissible if it is $\bar{\Gamma}_g(F)$-admissible for all $g \in \mathcal{C}$.

From such a fan, we obtain a partial toroidal compactification

$$\mathrm{Sh}_K(G,X)_\Delta = \cup(\Gamma_g \backslash X^+)_\Delta.$$

Definition 3.9. A family of fans (Δ^F), with w^F running over the admissible Cayley filtrations of G, is *K-admissible* if

(a) each Δ^F is $\bar{\Gamma}(F)$-admissible;

(b) for all $g \in \mathcal{C}$ and all $\gamma \in \Gamma_g$, we have $\gamma\Delta^F = \Delta^{\gamma F}$ where $W.^{\gamma F} =_{\mathrm{df}} \mathrm{ad}(\gamma) \cdot W.^F$;

(c) if $N^F \subset N^{F'}$, then $\Delta(F') = \{\sigma \cap C(F') \mid \sigma \in \Delta^F\}$.

A K-admissible family of fans $\Delta = (\Delta^F)$ defines a toroidal embedding $\mathrm{Sh}(G,X) \hookrightarrow \mathrm{Sh}(G,X)_\Delta$. We say that $\mathrm{Sh}(G,X)_\Delta$ is a *smooth toroidal compactification* if $\mathrm{Sh}(G,X)_\Delta$ is smooth and the boundary is a divisor with normal crossings, and we write $\mathrm{Sh}(G,X)^*$ for the projective system of smooth toroidal compactifications of $\mathrm{Sh}(G,X)$. The actions of $G(\mathbb{A}_f)$ and $\mathcal{G}(G)$ on $\mathrm{Sh}(G,X)$ extend to $\mathrm{Sh}(G,X)^*$.

Notes. There is a more detailed discussion, from a somewhat different point of view, of toroidal compactifications of nonconnected Shimura varieties in Harris (1989), §2.

5. Canonical models of toroidal compactifications.

Connected Shimura varieties. Let (G,X^+) be a pair defining a connected Shimura variety. Let x be a special point of X^+, and let τ be an automorphism of $\mathbb{C}$. Recall from (II.4.2) that there is a unique isomorphism

$$\varphi^0_{\tau,x} : \tau\mathrm{Sh}^0(G,X^+) \to \mathrm{Sh}^0({}^{\tau,x}G, {}^{\tau,x}X^+)$$

sending $\tau[x]$ to $[{}^\tau x]$ and such that $\varphi^0_{\tau,x} \circ \tau\mathcal{T}(g) = \mathcal{T}({}^{\tau,x}g) \circ \varphi^0_{\tau,x}$. It would be very surprising if the following statement were not true:

CONJECTURE 5.1. *The isomorphism $\varphi^0_{\tau,x}$ extends uniquely to an isomorphism*

$$\varphi^{0\,*}_{\tau,x} : \tau\mathrm{Sh}^0(G,X^+)^* \to \mathrm{Sh}^0({}^{\tau,x}G, {}^{\tau,x}X^+)^*.$$

Moreover, the following diagram commutes:

$$\begin{array}{ccc}
\varphi^{0\,*}_{\tau,x} : \tau\mathrm{Sh}^0(G,X^+)^* & \longrightarrow & \mathrm{Sh}^0({}^{\tau,x}G, {}^{\tau,x}X^+)^* \\
\downarrow & & \downarrow \\
\varphi^{0\,-}_{\tau,x} : \tau\mathrm{Sh}^0(G,X^+)^- & \longrightarrow & \mathrm{Sh}^0({}^{\tau,x}G, {}^{\tau,x}X^+)^-.
\end{array}$$

(The vertical arrows are the natural maps from the toroidal compactification to the minimal compactification.)

Note that, because $\mathrm{Sh}^0({}^{\tau,x}G, {}^{\tau,x}X^+)^*$ is separated and $\mathrm{Sh}^0(G,X^+)$ is dense in $\mathrm{Sh}^0(G,X^+)^*$, $\varphi^{0;*}_{\tau,x}$ will certainly be unique if it exists. It appears likely that the following argument will suffice to prove the existence of $\varphi^{0*}_{\tau,x}$. For connected Shimura varieties of Hodge type, the existence of $\varphi^{*}_{\tau,x}$ follows from the description of $\mathrm{Sh}^0(G,X^+)_\Delta$ as a moduli space of degenerating abelian varieties (see Chai and Faltings (1989) and Brylinski (1983), §4). To apply the strategy of II.9, the following statement will be needed:

(*) an inclusion $(G,X^+) \hookrightarrow (G',X'^+)$ induces a closed immersion
$$\mathrm{Sh}^0(G,X)^* \hookrightarrow \mathrm{Sh}^0(G',X'^+)^*.$$

Note that we already know that the map $\mathrm{Sh}^0(G,X^+) \hookrightarrow \mathrm{Sh}^0(G',X'^+)$ is a closed immersion (cf. Deligne 1971c, 1.15), and so (*) comes down to a combinatorial question about fans. Let $\Gamma \subset G^{\mathrm{ad}}(\mathbb{Q})^+$ and $\Gamma' \subset G'^{\mathrm{ad}}(\mathbb{Q})^+$ be such that $\Gamma\backslash X^+ \hookrightarrow \Gamma'\backslash X'^+$ is a closed immersion; when Δ is a Γ-admissible fan, we wish to find a Γ'-admissible fan Δ' such that the preceding map extends to a closed immersion $(\Gamma\backslash X^+)_\Delta \hookrightarrow (\Gamma'\backslash X'^+)_{\Delta'}$ (after possibly replacing Δ by a refinement Δ''). For this we can take Δ' to be any Γ'-admissible fan refining the image of Δ, and apply (Harris 1989, §3) to obtain a Δ'' for which $(\Gamma\backslash X^+) \hookrightarrow (\Gamma'\backslash X'^+)$ extends to a map $(\Gamma\backslash X^+)_{\Delta''} \hookrightarrow (\Gamma'\backslash X'^+)_{\Delta'}$.

Now assume $G = \mathrm{Res}_{L/\mathbb{Q}} G'$ with G' absolutely simple. After extending L we can suppose that there is an inclusion $(G_\alpha, X^+_\alpha) \hookrightarrow (G,X^+)$ with G_α of type A_1 and such that a boundary point of $\mathrm{Sh}^0(G_\alpha,X_\alpha)$ maps into any particular boundary component of $\mathrm{Sh}^0(G,X^+)^-$ we choose (see 2.6b). Then the domain of definition of the rational map
$$\tau\mathrm{Sh}^0(G,X^+)^* \to \mathrm{Sh}^0({}^{\tau,x}G, {}^{\tau,x}X.)^*$$
includes at least one point of the boundary component in question, and the Hecke operators then allow us to show that it will contain all points.

In practice, conjecture (5.1) is probably all one will need—in most situations where toroidal compactifications are needed, exactly which toroidal compactification is being used is irrelevant. In fact, the usual procedure is to choose a toroidal compactification and then show that the statements or objects one arrives at are independent of the choice. Nevertheless, it would be interesting to have a more precise result than (5.1) where, starting from a fan Δ, one constructs a fan Δ' for which $\varphi^0_{\tau,x}$ extends to an isomorphism
$$\tau\mathrm{Sh}^0(G,X)_\Delta \to \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)_{\Delta'}.$$

It is easy to guess what Δ' should be. For simplicity, assume G to be simply connected. Let F be a rational boundary component of X^+, and let Δ be a $\bar{\Gamma}(F)$-admissible fan. We wish to identify $\tau\mathrm{Sh}^0(G,X)_\Delta$ with a partial compactification of $\mathrm{Sh}^0({}^{\tau,x}G, {}^{\tau,x}X)$. Choose a faithful representation (V,ξ) of G^{ad}. Associated with this data, we have a one-motive $M = (X_M \to G_M)$ such that $U^F = \mathrm{Hom}(S^2X_M, \mathbb{C})$. The fan Δ corresponds to a torus embedding $T \hookrightarrow X_\Delta$ of $T = \mathrm{Hom}(S^2X_M, \mathbb{G}_m)$. Then τM is the motive attached to ${}^\tau x \in {}^{\tau,x}X$, and we can choose $\Delta' \subset \mathrm{Hom}(S^2X_{\tau M}, \mathbb{C})$ to be the fan corresponding to the torus embedding $\tau T \hookrightarrow \tau X_\Delta$.

CONJECTURE 5.2. *The isomorphism $\varphi^0_{\tau,x}$ extends uniquely to an isomorphism*

$$(\varphi^0_{\tau,x})_\Delta : \tau\mathrm{Sh}^0(G,X)_\Delta \to \mathrm{Sh}^0({}^{\tau,x}G, {}^{\tau,x}X)_{\Delta'}.$$

compatible with the maps to the minimal compactification.

Shimura varieties. Let (G,X) be a pair defining a Shimura variety.

THEOREM 5.3. *Assume (5.1). For any $\tau \in Aut(\mathbb{C})$ and special point $x \in X$, the isomorphism $\varphi_{\tau,x} : \tau\mathrm{Sh}(G,X) \to \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)$ of (II.4.2) extends uniquely to an isomorphism $\varphi^*_{\tau,x} : \tau\mathrm{Sh}(G,X)^* \to \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X^+)^*$; moreover, the diagram*

$$\begin{array}{ccc} \tau\mathrm{Sh}(G,X)^* & \longrightarrow & \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)^* \\ \downarrow & & \downarrow \\ \tau\mathrm{Sh}(G,X)^- & \longrightarrow & \mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)^- \end{array}$$

commutes.

PROOF: This can be obtained by induction from (5.1).

COROLLARY 5.4. *(Assuming 5.1.)*

(a) $\mathrm{Sh}(G,X)^*$ *has a canonical model over* $E(G,X)$.

(b) For any $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, $\tau\mathrm{Sh}(G,X)^*$ *is canonically isomorphic to the canonical model of* $\mathrm{Sh}({}^{\tau,x}G, {}^{\tau,x}X)^*$ *over* $\tau E(G,X)$.

Conjecture 5.2 has an obvious analogue for nonconnected Shimura varieties.

Remark 5.5. So far we have not mentioned Eisenstein series. Briefly, Eisenstein series attach an automorphic form on the whole Shimura variety to a cusp form on a boundary component. This construction should be compatible with all the isomorphisms in this article. In particular, an Eisenstein series should be defined over a field E when the cusp form is.

Notes. As we noted (3.7a) a smooth projective toroidal compactification is obtained from the minimal compactification by blowing it up at certain ideals, described by the polarizing function f, and then normalizing. Brylinski (1983) uses this and Fourier-Jacobi series to prove the existence of canonical models of projective toroidal compactifications of Shimura varieties of Hodge type. Harris (1989), 2.8, suggests that the results in Harris (1986) can be used to generalize this result. (I understand that Richard Pink will also examine the question of the existence of canonical models of toroidal compactifications in his Bonn thesis.)

6. Canonical extensions of automorphic vector bundles. First we note that automorphic vector bundles extend to toroidal compactifications.

THEOREM 6.1. *Let (G,X) be a pair defining a Shimura variety, and let $\mathrm{Sh}(G,X)_\Delta$ be a smooth toroidal compactification of $\mathrm{Sh}(G,X)$. There is an exact faithful functor $\mathcal{J} \mapsto \mathcal{V}_\Delta(\mathcal{J})$ from the category of $G^c_{\mathbf{C}}$-vector bundles on $\check{X}$ to that of vector bundles on $\mathrm{Sh}(G,X)_\Delta$ such that*

(a) $\mathcal{V}(\mathcal{J})_\Delta|\mathrm{Sh}(G,X) = \mathcal{V}(\mathcal{J})$ (notation as in III.2);

(b) $\mathcal{J} \mapsto \mathcal{V}(\mathcal{J})_\Delta$ commutes with tensor products and duals (i.e., it is a morphism of tensor categories);

(c) equivariant differential operators between the $\mathcal{V}(\mathcal{J})$'s extend to the $\mathcal{V}_\Delta(\mathcal{J})$'s.

Moreover, $\mathcal{V}_\Delta(\mathcal{J})$ is uniquely determined by the properties (a), (b), and (c).

PROOF: Let $o \in \check{X}$. Then $G_{\mathbf{C}}$-vector bundles correspond to representations of P_o (see III.2.3a). For the $\mathcal{J}$ corresponding to irreducible representations of P_o, the result is essentially proved in Mumford (1977).

For Siegel modular varieties, the theorem is proved in Chai and Faltings (1989), VI.4. Briefly, their proof proceeds as follows. Using the structure of $\mathrm{Sh}(G,X)$ near the boundary, it is possible to construct the extension of $\mathcal{V}(\mathcal{J})_\Delta$ locally; the problem is to show that the local extensions patch. It is clear that the category of representations of P_o for which this is true is closed under tensor products, duals, and subquotients. It therefore suffices to construct $\mathcal{V}(\mathcal{J}_o)_\Delta$ for a single faithful representation of P_o. Chai and Faltings take $\mathcal{J}_o$ to be the standard representation G, and show that $\mathcal{V}(\mathcal{J}_o)_\Delta$ can be obtained from the de Rham cohomology of the universal semi-abelian scheme that they have already constructed. Harris (1989) shows that by applying Deligne's existence theorem (Deligne 1970) it is possible to avoid using the universal semi-abelian scheme.

THEOREM 6.2. *Let* $\mathrm{Sh}(G,X)_\Delta$ *be a smooth toroidal compactification of a Shimura variety having a canonical model over* $E \supset E(G,X)$. *If* $\mathcal{J}$ is defined over E, then so also is $\mathcal{V}(\mathcal{J})_\Delta$.

PROOF: The descent datum on $\mathcal{V}(\mathcal{J})$ extends to $\mathcal{V}(\mathcal{J})_\Delta$.

Once general results on canonical models have been obtained, essentially all the results in Chapter III will extend to vector bundles on the toroidal compactifications.

Notes. See the references in the proof of (6.1).

VI. MIXED SHIMURA VARIETIES

In this chapter, we suggest how the results of Chapter II should generalize to mixed Shimura varieties.

1. Definition of a mixed Shimura variety.

Let P be a connected algebraic group over $\mathbb{Q}$. Recall from (I.1) that we have the notion of a filtration $W.$ of $\mathbf{Rep}_{\mathbb{Q}}(P)$. Moreover, $P = W_0 P$ if and only if P preserves the filtration on each representation of P, and $W_{-1}P$ is the (unipotent) subgroup of W_0P acting trivially on $\mathrm{Gr}^W_\cdot(V)$ for all representations of P. For any cocharacter w of P splitting the filtration, $W_0 P = W_{-1} \rtimes Z(w)$, where $Z(w)$ is centralizer of w.

The axioms for a mixed Shimura variety. The datum needed to define a mixed Shimura variety is a triple $(P, W., Y)$ comprising a connected algebraic group P over $\mathbb{Q}$, an ascending filtration $W.$ of $\mathbf{Rep}_{\mathbb{C}}(P)$, and a $P(\mathbb{R}) \cdot (W_{-2}P(\mathbb{C}))$-conjugacy class Y of descending filtrations of $\mathbf{Rep}_{\mathbb{C}}(P)$. For $y \in Y$, write $F_y^\cdot$ for the filtration defined by $y \in Y$. The filtration $W.$ is defined over some totally real number field, and the filtration it induces on $\mathbf{Rep}_{\mathbb{C}}(P/Z(P))$ is defined over $\mathbb{Q}$. The triple is required to satisfy the following conditions:

(1.1.0) for any representation (ξ, V) of P, $W.$ and $F_y^\cdot$ define a real mixed Hodge structure on $V(\mathbb{R})$, all $y \in Y$;

(1.1.1) $\mathrm{Lie}(P_{\mathbb{C}}) = W_0\mathrm{Lie}(P_{\mathbb{C}}) = F_y^{-1}\mathrm{Lie}(P_{\mathbb{C}})$ for each $y \in Y$;

(1.1.2) for any μ_y splitting the filtration $F_y^{\cdot}$, $\mu_y(i)\cdot\overline{\mu_y(\mathrm{i})}$ is a Cartan involution on $(\mathrm{Gr}_0^W P)^{\mathrm{ad}}$;

(1.1.3) $(\mathrm{Gr}_0^W P)^{\mathrm{ad}}$ has no $\mathbb{Q}$-rational factors that are anisotropic over $\mathbb{R}$;

(1.1.4) $Z(P)^0$ is a torus, splitting over a CM-field;

(1.1.5) the (adjoint) action of $\mathrm{Gr}_0^W P$ on $\mathrm{Gr}_{-1}^W \mathrm{Lie}\, P$ factors through $\mathrm{Gr}_0^W P)^c$ (notation as in the introduction to Chapter III).

Simplifications occur when we strengthen some of the axioms:

(1.1.0*) the filtration $W.$ is defined over $\mathbb{Q}$, and $W.$ and $F_y^{\cdot}$ define a rational mixed Hodge structure on V for any representation (V, ξ) of P;

(1.1.2*) for any μ_y splitting the filtration $F_y^{\cdot}$, $\mu_y(i)\cdot\overline{\mu_y(i)}$ is a Cartan involution on $P/(W_{-1}P \cdot w(\mathbb{G}_m))$;

(1.1.4*) (1.1.4) holds and there is a one-dimensional representation V_0 of P such that $(V_0, W., F_y^{\cdot})$ is the pure Hodge structure $\mathbb{Q}(1)$ for all y.

We usually drop the W from the notation Gr_r^W. For each $y \in Y$, there is a homomorphism $\tilde{h}_y : \mathbb{G}_m \times \mathbb{G}_m \to P_{\mathbb{C}}$ such that, for every representation (V, ξ) of P, $\xi \circ \tilde{h}_y$ provides $V(\mathbb{C})$ with the bigrading associated with the mixed Hodge structure (see IV.1). It is important to note, however, that in general $\tilde{h}_{py} \neq (\mathrm{ad}\, p) \circ \tilde{h}_y$ unless $p \in P(\mathbb{R})$.

Remark 1.2. (a) Axiom (1.1.5) has been imposed only so that the mixed Shimura variety exists as a scheme rather than a stack. Probably this condition should be dropped. In any case, the axioms should be viewed as tentative.

(b) Axiom (1.1.2) implies that $(\mathrm{Gr}_0 P)^{\mathrm{ad}}$ is semisimple, and (1.1.4) implies that the connected centre of $\mathrm{Gr}_0 P$ is a torus. Therefore $\mathrm{Gr}_0 P$ is a reductive group, $W_{-1}P_{\mathbb{C}}$ is the unipotent radical of $P_{\mathbb{C}}$, and, for any w splitting $W.$, $Z(w)$ is a Levi subgroup of $P_{\mathbb{C}}$. Note that if $\mathrm{Gr}_0 P = 0$, then $w(\mathbb{G}_m) = 0$, which implies that $W_{-1}P = 0$ and that $P = 0$.

(c) Let $\mathrm{Lie}(P)_{\mathbb{C}} = \oplus \tilde{H}^{p,q}$ be the decomposition of $\mathrm{Lie}(P)_{\mathbb{C}}$ corresponding to the mixed Hodge structure $(W., F_y^{\cdot})$ some $y \in Y$. Then (1.1.1) implies that $\tilde{H}^{p,q} = 0$ for $p + q > 0$ and $p < -1$. Hence

$\mathrm{Gr}_0(\mathrm{Lie}\, P)$ has a Hodge structure of type $\{(-1,1), (0,0), (1,-1)\}$;

$\mathrm{Gr}_{-1}(\mathrm{Lie}\, P)$ has a Hodge structure of type $\{(-1,0), (0,-1)\}$;

$\mathrm{Gr}_{-2}(\mathrm{Lie}\, P)$ has a Hodge structure of type $\{(0,0)\}$;

(see the picture in IV.2.10). Thus

$$\text{(1.2.1)} \qquad \operatorname{Lie} P_{\mathbb{C}} = \operatorname{Lie} P_{\mathbb{R}} + F_y^0 \operatorname{Lie} P_{\mathbb{C}} + W_{-2} \operatorname{Lie} P_{\mathbb{C}}.$$

From the last equality it follows that Y can also be regarded as a $P(\mathbb{R}) \cdot W_{-1}P(\mathbb{C})$-conjugacy class.

(d) It suffices to check (1.1.0) for a single $y \in Y$ (cf. Brylinski 1983, 2.3.1.2), and for a finite family of representations (V_i, ξ_i) such that $\cap \operatorname{Ker}(\xi_i)$ is finite (see IV.1.3).

The complex structure on Y.

PROPOSITION 1.3. *Let $\check{Y}$ be the $P(\mathbb{C})$-conjugacy class of filtrations of $\mathbf{Rep}_{\mathbb{C}}(P)$ containing $F_y^{\cdot}$ for all $y \in Y$. Then $\check{Y}$ is a Grassman variety, and the map*

$$\beta : Y \hookrightarrow \check{Y},\ y \mapsto F_y^{\cdot},$$

identifies Y with an open complex submanifold of $\check{Y}$. The induced complex structure on Y is the unique structure such that, for all representations (V, ξ) of P, the filtrations $F_y^{\cdot}$ on $\mathcal{V}(\xi) =_{\text{df}} Y \times V(\mathbb{C})$ vary homolomorphically.

PROOF: Fix a point $o \in Y$. Then $\check{Y} = P(\mathbb{C})/F_o^0P(\mathbb{C})$, which is a Grassman variety, and β is the map

$$g \cdot o \mapsto g \pmod{F_o^0 P(\mathbb{C})} : Y \to P(\mathbb{C})/F_o^0P(\mathbb{C}).$$

This is obviously injective, and (1.2.1) shows that it identifies Y with an open (almost) complex submanifold $\check{Y}$.

PROPOSITION 1.4. *Let $\xi : P \to GL(V)$ be a rational representation of P, and let $V_{\mathbb{R}} = \oplus V(i)$ be the decomposition of $V_{\mathbb{R}}$ under the action of $Z(P)^0_{\mathbb{R}}$; then $y \mapsto (V(i), W_{\cdot}, F_y^{\cdot})$ is a variation of real mixed Hodge structures on Y.*

PROOF: On $\operatorname{Gr}_n(V(i))$, we have a representation of $\operatorname{Gr}_0(P)$; apply (II.3.2) to see that it defines a variation of real Hodge structures. The transversality axiom (condition (H_1) of (IV.1)) follows from the fact that $\operatorname{Lie} P_{\mathbb{C}} = F_y^{-1}(\operatorname{Lie} P_{\mathbb{C}})$.

Define Y' to be the $(P/W_{-2}P)(\mathbb{R})$-conjugacy class of filtrations of $\mathbf{Rep}_{\mathbb{C}}(P/W_{-2}P)$ containing the image of Y, and define X to be the $(\operatorname{Gr}_0 P)(\mathbb{R})$-conjugacy class of filtrations of $\mathbf{Rep}_{\mathbb{C}}(\operatorname{Gr}_0 P)$ containing the image of Y'. Proposition 1.3 shows that both Y' and X also have natural complex structures.

PROPOSITION 1.5. *The natural maps* $Y \xrightarrow{\pi_1} Y' \xrightarrow{\pi_2} X$ *are both holomorphic. Moreover,*

X is a symmetric Hermitian domain;
$Y' \to X$ is a fibre bundle with structure group $V(\mathbb{R})$, $V = Gr_{-1}(P)$;
$Y \to Y'$ is a fibre bundle with structure group $U(\mathbb{C})$, $U = Gr_{-2}(P)$.

PROOF: Straightforward from the definitions (and II.3.2).

Write π for the composite $Y \to X$.

The mixed Shimura variety. For any compact open subgroup K of $P(\mathbb{A}_f)$, define

$$M_K(P, W., Y) = P(\mathbb{Q})\backslash Y \times P(\mathbb{A}_f)/K.$$

It is a complex manifold if K is sufficiently small; in fact, it is a disjoint union of varieties of the form $\Gamma\backslash Y^+$ with Y^+ a connected component of Y and Γ a discrete subgroup of $P(\mathbb{R})^+$. Each $g \in P(\mathbb{A}_f)$ defines a holomorphic map,

$$\mathcal{T}(g) : M_K(P, W., Y) \to M_{g^{-1}Kg}(P, W., Y),\ [y, p] \mapsto [y, pg].$$

THEOREM 1.6. *(a) The complex manifold $M_K(P, W., Y)$ has a natural structure as an algebraic variety. More precisely, it is a torus bundle over a polarizable abelian scheme over a Shimura variety.*

(b) For each $g \in P(\mathbb{A}_f)$, $\mathcal{T}(g)$ is algebraic.

PROOF: For any quotient P' of P by a subgroup of $Z(P)$, we have a triple $(P', W.', Y')$ satisfying the axioms (1.1), and for each pair of open compact subgroups $K \subset P(\mathbb{A}_f)$ and $K' \subset P'(\mathbb{A}_f)$ such that K' contains the image of K, there is a morphism

$$M_K(P, W., Y) \to M_{K'}(P', W.', Y').$$

Each connected component of $M_K(P, W., Y)$ is a finite covering of a connected component of $M_{K'}(P', W.', Y')$. Thus, if we can prove (a) for $(P', W.', Y')$, then the Riemann existence theorem will show that it is also true for $(P, W., Y)$. A similar remark applies to (b). This allows us to assume that conditions (1.1.0*) and (1.1.2*) hold. Later in this section we outline a proof of the theorem in this case.

We obtain a scheme $M(P, W., Y)$ with a continuous action of $P(\mathbb{A}_f)$, which we call the *mixed Shimura variety* defined by $(P, W., Y)$.

Special points. A point $y \in Y$ is said to be *special* if for one faithful (hence every) representation (V, ξ) of P^{ad}, the mixed Hodge structure $(V, W_\cdot, F_y^\cdot)$ decomposes into a sum of pure Hodge structures, each of CM-type. We say that y is a CM-point if the same condition holds for the representations of P itself. A mixed Hodge structure is said to be *rationally decomposed* if it is a direct sum of pure Hodge structures.

PROPOSITION 1.7. *(a) Let $x = \pi(y)$; then y is special if and only if x is special and $(V, W_\cdot, F_y^\cdot)$ is rationally decomposed for each representation of P^{ad}.*

(b) For each special $x \in X$, there is a $y \in \pi^{-1}(X)$ such that $(V, W_\cdot, F_y^\cdot)$ is rationally decomposed for each representation of P^{ad}.

PROOF: Part (a) is obvious. We outline a proof of (b) later in this section.

For each special point y, there is a unique homomorhism $\rho_y : \mathfrak{S} \to P^{\mathrm{ad}}$ such that $\rho_y \circ \mu_{\mathrm{can}} = \mu_y$. When y is a CM-point, ρ_y is a homomorphism $\mathfrak{S} \to P$.

Connected mixed Shimura varieties. Let $(P, W_\cdot, Y)$ define a mixed Shimura variety, and let $(G, X) = (\mathrm{Gr}_0 P, Y \pmod{W_{-1}P})$. The fibres of the map $M(P, W_\cdot, Y) \to \mathrm{Sh}(G, X)$ are connected, and so the inverse image of $\mathrm{Sh}(G, X)^0$ is connected. Let P' be the inverse image of G^{der} in P, let $W'_\cdot$ be the filtration of $\mathbf{Rep}(P^{\mathrm{ad}})$ defined by $W_\cdot$, and let Y^+ be a connected component of Y. Assume G^{der} to be simply connected. Then

$$M(P, W_\cdot, Y)^0 = \varprojlim P'(\mathbb{Q})\backslash Y^+ \times P'(\mathbb{A}_f)/K'.$$

In particular, $M(P, W_\cdot, Y)^0$ depends only on $(P', W'_\cdot, Y^+)$. Just as in the case of (pure) Shimura varieties, there is a theory of connected mixed Shimura varieties, which we will not discuss this further.

Examples. Mixed Shimura varieties abound.

Example 1.8. ($W_{-1}P = 0$; Shimura varieties). Let (G, X) be a pair satisfying (II.2.1). Set

$$P = G; \quad W_\cdot = \mathrm{Filt}(w_X); \quad Y = \{\mathrm{Filt}(\mu_x) \mid x \in X\}.$$

The triple $(P, W_\cdot, Y)$ satisfies the axioms (1.1) (use II.3.2), and the variety $M(P, W_\cdot, Y) = \mathrm{Sh}(G, X)$. Conversely, if $(P, W_\cdot, Y)$ satisfies

(1.1) and $W_{-1}P = 0$, then P is a reductive group and the pair (P, X), $X = \{z \mapsto \tilde{h}_y(z, \bar{z}) \mid y \in Y\}$ satisfies the axioms (II.2.1). Thus mixed Shimura varieties defined by triples $(P, W., Y)$ with $W_{-1}P = 0$ are Shimura varieties, and every Shimura variety is of this form.

Example 1.9. ($\mathrm{Gr}_{-1}P = 0$; automorphic vector bundles). Consider a triple $(P, W., Y)$ satisfying (1.1) and (1.1.0*), and assume that $\mathrm{Gr}_{-1}P = 0$. Write $U = W_{-2}P$. It is commutative, and so the exponential map allows us to identify it with its Lie algebra. The adjoint action defines a representation ξ of P on U, factoring through $G =_{\mathrm{df}} \mathrm{Gr}_0 P$. Then $M_K(P, W., Y) = \mathcal{V}_K(\xi)/(\text{lattice})$, where $\mathcal{V}_K(\xi)$ is the automorphic vector bundle on $\mathrm{Sh}_K(G, X)$ defined by (V, ξ). The fibre of $M_K(P, W., Y)$ over a point of $\mathrm{Sh}_K(G, X)$ is $V(\mathbb{C})/\Lambda$ for some lattice Λ in V, and the exponential map shows that this is isomorphic to a product of copies of $\mathbb{C}^\times$. In particular, $M(P, W., Y)$ is algebraic (by III.2.1).

Conversely, let (G, X) be a pair satisfying (II.2.1*), and let (U, ξ) be a faithful representation of G. Define $P = U \rtimes G = \left\{\begin{pmatrix} 1 & 0 \\ u & g \end{pmatrix}\right\}$, and let it act on $V =_{\mathrm{df}} U \otimes U$ in the obvious way. Define a filtration of V by

$$0 = W_{-3}V \subset U \oplus 0 = W_{-2}V \subset V = W_0 V,$$

and give P the induced filtration $W.$. Define Y to be the set of filtrations of $\mathbf{Rep}_{\mathbb{C}}(P)$ inducing on $\mathbf{Rep}_{\mathbb{C}}(G)$ the Hodge filtration corresponding to some $x \in X$. Then $(P, W., Y)$ defines a mixed Shimura variety, which is a quotient of the automorphic vector bundle $\mathcal{V}(\xi)$ on $\mathrm{Sh}(G, X)$.

Example 1.10. ($W_{-2}P = 0$; Kuga varieties). Consider a triple $(P, W., Y)$ satisfying (1.1) and (1.1.0*), and assume that $\mathrm{Gr}_{-2}P = 0$. Write $V = W_{-1}P$. It is a commutative algebraic group over $\mathbb{Q}$, and so the exponential map allows us to regard it as a vector space. The adjoint action defines a representation ξ of P on V, factoring through $G =_{df} Gr_0 P$. Each $y \in Y$ defines a Hodge structure on V of type $\{(-1, 0), (0, -1)\}$, which, according to (II.3.2), is polarizable. The choice of a compact open subgroup K' of $P(\mathbb{A}_f)$ defines a lattice in V, and consequently we obtain a family of abelian varieties $\mathcal{A}$ over $\mathrm{Sh}_K(G, X)$, where K is the image of K' in $G(\mathbb{A}_f)$ (cf. II.3.11). We have $M_{K'}(P, W., Y) = \mathcal{A}$. In particular, $M(P, W., Y)$ is algebraic.

The simplest example of such a mixed Shimura variety is the universal elliptic curve over $\mathrm{Sh}(GL_2, X)$. This (rather, a connected component of it) has been extensively studied; see for example Eichler and Zagier (1985) and Berndt (1983).

A more interesting case is that where the base Shimura variety is defined by a quaternion algebra over a totally real field (not necessarily totally indefinite, so the Shimura variety is not a moduli variety; see Deligne 1979, §6, Modèles étranges). These mixed Shimura varieties (rather, their connected components) have been extensively studied by students of Kuga; see for example Addington (1987) and Petri (1989).

We have noted that a connected component of a mixed Shimura varieties with $W_{-2}P = 0$ is a Kuga fibre variety, but the converse is not true: there are "nonrigid" Kuga fibre varieties that move in families and do not have models over number fields.

Example 1.11. (Mixed Shimura varieties arising from boundary components). Consider a Shimura variety $\mathrm{Sh}(G, X)$, and let $W.$ be an admissible Cayley filtration of G (see V.4.2). Define P to be the subgroup of $W_0 G$ acting trivially on $U =_{\mathrm{df}} W_{-2}G$. Then there is a natural way to attach to $W.$ a family Y of filtrations of $\mathbf{Rep}_{\mathbb{C}}(P)$ so that $(P, W., Y)$ defines a mixed Shimura variety. The base Shimura variety is $\mathrm{Sh}(\mathrm{Gr}_0(P), F)$, where F is the rational boundary component of X corresponding to $W.$.

Example 1.12. (Mixed Shimura varieties of Hodge type). Let M be a one-motive over $\mathbb{Q}$, and let P be the Mumford-Tate group of M. The weight and Hodge filtrations on $H_B(M)$ define filtrations $W.$ and $F_o^{\cdot}$ on $\mathbf{Rep}_{\mathbb{C}}(P)$. Let Y be the $P(\mathbb{R}) \cdot W_{-2}P(\mathbb{C})$-conjugacy class of $F_o^{\cdot}$. Then $(P, W., Y)$ satisfies the stronger axioms (1.1*) (see IV.2.9). A mixed Shimura variety $M(P, W., Y)$ will be said to be of *Hodge type* if there is a one-motive M and a representation (V, ξ) of P such that

(a) for some $o \in Y$, $(H_B(M), W., F^{\cdot}) = (V, W., F_o^{\cdot})$;

(b) P is the subgroup of $GL(H_B(M)) \times \mathbb{G}_m$ fixing a family of Hodge tensors.

Such a mixed Shimura variety is a (coarse) moduli variety for a family of one-motives with Hodge cycle and level structures. Note that the total space of a fine moduli variety for abelian varieties is a moduli variety for one-motives of the form $(\mathbb{Z} \to A)$.

Outline for proofs of 1.6 and 1.7. Since it suffices to prove both statements for a triple $(P, W., Y)$ satisfying (1.1.0*) and (1.1.2*), we henceforth assume this. We have already verified the statements when:

(i) $P = \mathrm{Gr}_0(P)$; then $M(P, W., Y)$ is a Shimura variety;

(ii) $W_{-2}P = 0$; then $M(P, W., Y)$ is the total space of an abelian scheme over $\mathrm{Sh}(G, X)$;

(iii) $\mathrm{Gr}_{-1}(P) = 0$; then both statements reduce to statements about automorphic vector bundles.

The next lemma is slightly stronger than (1.7b).

LEMMA 1.13. *Let $(P, W., Y)$ be as above, and let $x \in X$. For every representation (V, ξ) of P, there exists a $y \in \pi^{-1}(x)$ such that the mixed Hodge structure $(V, W., F^{\cdot}_y)$ is rationally decomposed.*

PROOF: Fix a $y \in \pi^{-1}(x)$. We have to show that there is a $p \in W_{-1}P(\mathbb{C})$ such that $(V, W., pF^{\cdot}_y)$ is rationally decomposed. The proof proceeds by induction on the length of the filtration $W.$ of V (see Brylinski 1983, 2.3.1.5).

Under our assumptions, a representation (V, ξ) of P defines a variation of mixed Hodge structures $\mathcal{V}$ on $M(P, W., Y)$. Let K be a compact open subgroup of $P(\mathbb{A}_f)$, and write K also for its image in $G(\mathbb{A}_f)$, $G = \mathrm{Gr}_0 P$.

LEMMA 1.14. *There exists a section $s : \mathrm{Sh}_K(G, X) \to M_K(P, W., Y)$ to π such that $s^*(\mathcal{V})$ is rationally decomposed (after possibly replacing K by a subgroup).*

PROOF: See Brylinksi (1983), 2.3.1.7.

Thus we get a canonical section $s : \mathrm{Sh}(G, X) \to M(P, W., Y)$ to π.

We now come to the proof of (1.6). First, the sheaf $R\pi_{1*}\mathbb{Z}$ is constant. Thus it splits up (analytically) under the characters of T, where T is the algebraic torus $W_{-2}P(\mathbb{C})/W_{-2}\Gamma$, $W_{-2}\Gamma = K \cap (W_{-2}P(\mathbb{C}))$. Let ρ be such a character.

LEMMA 1.15. *There exists on each $\mathcal{L}_\rho$ a unique algebraic structure such that*

(i) $\mathcal{L}_{2\rho}$ is isomorphic (algebraically) to $\sigma^*(\mathcal{L}_{2\rho})$ (σ is the map $x \mapsto -x$ on $\mathcal{A}$);

(ii) The restriction of $\mathcal{L}_\rho$ to the zero section of $\mathcal{A}$ is trivial.

Moreover, $\mathcal{L}_\rho|$(zero section) is canonically trivial.

PROOF: Brylinski (1983), 2.3.2.4.

LEMMA 1.16. *$M(P, W., Y)$ has a unique algebraic structure such that*

(i) σ: $\mathcal{A} \to \mathcal{A}$ is algebraic;

(ii) the section $s : \mathrm{Sh}(G,X) \to M(P,W.,Y)$ is algebraic.

PROOF: See Brylinski (1983), 2.3.2.5.

Remark 1.17. If $M(P,W.,Y)$ is of Hodge type, then a representation of P defines an algebraic family of one-motives over $M(P,W.,Y)$, except that the family may only exist as a stack (cf. III.8).

Notes. The general notion of a mixed Shimura variety is due to Deligne. A slightly restricted form can be found in Brylinski's thesis (Brylinski 1983), where the varieties are called generalized Shimura varieties. The proofs of (1.6) and (1.7) are adapted from this source.

2. Canonical models of mixed Shimura varieties. Let y be a special point of Y. Then we get a homomorphism $\rho_y : \mathfrak{S} \to P^{\mathrm{ad}}$. Thus the $\mathfrak{S}$-torsor ${}^\tau\mathfrak{S}$ can be used to twist P to give a group ${}^{\tau,y}P$, and the canonical element $\mathrm{sp}(\tau)$ defines an isomorphism $g \mapsto {}^{\tau,y}g : P(\mathbb{A}_f) \to {}^{\tau,y}P(\mathbb{A}_f)$. Define ${}^{\tau,y}Y$ to be the conjugacy class containing $\tau F_y^{\cdot}$ for y a special point of Y. Then the triple $({}^{\tau,y}P, {}^{\tau,y}W., {}^{\tau,y}Y)$ satisfies the axioms for a mixed Shimura variety.

CONJECTURE 2.1. *For each $\tau \in Aut(\mathbb{C})$, there exists a unique isomorphism*

$$\varphi_{\tau,y} : \tau M(P,W.,Y) \to M({}^{\tau,y}P, {}^{\tau,y}W., {}^{\tau,y}Y)$$

such that
(i) $\varphi_{\tau,y}(\tau[y,1]) = [{}^\tau y, 1]$;
(ii) $\varphi_{\tau,y} \circ \tau\mathcal{T}(g) = \mathcal{T}({}^{\tau,y}g) \circ \varphi_{\tau,y}$ for all $g \in G(\mathbb{A}_f)$.
Moreover, when y' is a second special point in Y, then there is a canonical map

$$\varphi(\tau; y', y) : M({}^{\tau,y}P, {}^{\tau,y}W., {}^{\tau,y}Y) \to M({}^{\tau,y'}P, {}^{\tau,y'}W., {}^{\tau,y'}Y),$$

and we have the identity

$$\varphi(\tau; y', y) \circ \varphi_{\tau,y} = \varphi_{\tau,y'}.$$

Remark 2.2. We know the above result in several cases:

(i) $W_{-1}P = 0$. Here the mixed Shimura variety is a (pure) Shimura variety, and the conjecture is (II.4.2) and (II.4.4).

(ii) $\mathrm{Gr}_{-1}P = 0$. Here the conjecture follows from the results on automorphic vector bundles in Chapter III.

(iii) $W_{-2}P = 0$; assume (1.1.0*). Here the mixed Shimura variety is an abelian scheme over a Shimura variety. To give an abelian scheme over $\mathrm{Sh}(G, X)$ is the same as to give a polarizable variation of integral Hodge structures on $\mathrm{Sh}(G, X)$. In this case the conjecture follows from (III.6.2).

(iv) Mixed Shimura varieties of Hodge type. Here the conjecture follows from the fact that the mixed Shimura variety is a moduli variety for one-motives (see Brylinski 1983, 2.3.3.1).

Thus to complete the proof of the conjecture, it remains

(i) to lift the isomorphism

$$\tau M(P/W_{-2}P, W_\cdot, Y') \to M({}^{\tau,y}(P/W_{-2}P, {}^{\tau,y}W_\cdot, {}^{\tau,y}Y')$$

to the covering $\tau M(P, W_\cdot, Y)$ (equivalently, to the sheaves $\tau\mathcal{L}_\rho$ on $\tau M(P, W_\cdot, Y)$) in the case that (1.1.0*) holds, and

(ii) to remove the condition (1.1.0*).

Probably the best approach to (i) will be to deduce it from an extension of the theorems in Chapter III to automorphic vector bundles on mixed Shimura varieties (see §4 below). It should be possible to prove (ii) by using connected mixed Shimura varieties.

Just as for Shimura varieties, the conjecture will imply that a mixed Shimura variety has a canonical model over a reflex field (suitably defined), and that the conjugate of a canonical model by $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ is the canonical model of the mixed Shimura variety defined by the conjugate data.

3. Partial compactification of mixed Shimura varieties. Consider a mixed Shimura variety $M_K(P, W_\cdot, Y)$. Let $U = W_{-2}P$, and let T be the torus $U(\mathbb{C})/U(\mathbb{Z})$, where $U(\mathbb{Z}) = U(\mathbb{Q}) \cap K$. For a fan $\Delta \subset X_*(T) \otimes \mathbb{R} = U(\mathbb{R})$ satisfying suitable conditions, the construction in Chapter V can be mimicked to give a partial compactification

$$\pi_{1\Delta} : M(P, W_\cdot, Y)_\Delta \to M(P/W_{-2}P, W_\cdot, Y')$$

of the map $\pi_1 : M_K(P, W_\cdot, Y) \to M(P/W_{-2}P, W_\cdot, Y')$ (cf. Brylinski (1983), §4)). The isomorphism in (2.1) should extend to an isomorphism

$$\tau M(P, W_\cdot, Y)_\Delta \to M({}^{\tau,y}P, {}^{\tau,y}W_\cdot, {}^{\tau,y}Y)_{\Delta'}$$

for a suitable fan Δ' in ${}^{\tau,y}U(\mathbb{R})$.

4. Automorphic vector bundles.
As we saw in (1.3), there is an embedding $\beta : Y \hookrightarrow \check{Y}$ from Y into a variety of filtrations of $\mathbf{Rep}(P_{\mathbb{C}})$, and the action of $P_{\mathbb{C}}$ on $\check{Y}$ extends that of $P(\mathbb{R}) \cdot W_{-2}P(\mathbb{C})$ on Y. Let $\mathcal{J}$ be an $P_{\mathbb{C}}$-vector bundle on $\check{Y}$. If $\beta^*(\mathcal{J})$ defines a vector bundle $\mathcal{V}_K(\mathcal{J})$ on the quotient $M_K(P, W_., Y)$ of Y, then we call $\mathcal{V}_K(\mathcal{J})$ an *automorphic vector bundle* . The theorems in Chapter III for automorphic vector bundles on Shimura varieties should extend to mixed Shimura varieties.

5. Toroidal compactification of mixed Shimura varieties.
Consider a mixed Shimura variety,

$$M(P, W_., Y) \xrightarrow{\pi_1} M(P/W_{-2}P, W_., Y') \xrightarrow{\pi_2} \mathrm{Sh}(G, X).$$

Form a toroidal compactification $\mathrm{Sh}(G, X)_\Delta$ of $\mathrm{Sh}(G, X)$. It should be possible to compactify successively the morphisms π_2 and π_1. The compactifications of the total space of the Siegel modular variety by Namikawa over $\mathbb{C}$ (Namikawa 1976, 1979) and Chai over $\mathbb{Z}$ (Chai and Faltings 1989), should serve as models for the compactification π_2.

VII. Fourier-Jacobi series

Fourier-Jacobi series play a central role in the theory of holomorphic automorphic forms. In this chapter, we briefly indicate how they fit into the schema described in the first six chapters.

For elliptic modular forms, there are three different approaches to defining Fourier series: the (classical) analytic approach; the modular approach, based on the moduli of elliptic curves; and the formal-algebraic approach, based on analyzing the structure of the elliptic modular curve at its cusps. The first is available for a general Shimura variety, but is badly adapted for studying rationality questions. The second applies only to Shimura varieties of Hodge type. Therefore, it is the third approach that will be most important.

The q-expansion principal asserts that an automorphic form is determined by (certain of) its Fourier-Jacobi series. Since there should be the notion of the conjugate of a Fourier-Jacobi series by an automorphism of $\mathbb{C}$, and hence the notion of a Fourier-Jacobi series being rational over a field, this means that it will be possible to read off the field of rationality of an automorphic form from the coefficients of its Fourier-Jacobi series. Since these live on lower dimensional (mixed Shimura) varieties, this will be a useful tool.

1. Elliptic modular forms.

An elliptic modular function f of level N satisfies

$$f(z+N)=f(z), \quad z\in H^{+}.$$

It therefore has a Fourier expansion

$$f(z)=\sum a_n q_N^n, \quad q_N=e^{2\pi iz/N}$$

corresponding to the cusp at infinity, and a similar expansion at the other cusps. It is known that f is rational over a subfield $L\subset\mathbb{C}$ (in the sense of Chapter III) if and only if the coefficients of these series lie in L.

We next explain the moduli definition (for details, see Katz 1973). Let

$$K_N=\{\alpha\in GL_2(\hat{\mathbb{Z}})\mid \alpha\equiv I(\mathrm{mod} N)\}.$$

Write S_N for the corresponding modular curve $\mathrm{Sh}_{K(N)}(GL_2,H^{+})$, and $\mathcal{A}$ for the universal elliptic curve over S_N. On S_N we have the line bundle $\omega=\omega_{\mathcal{A}/S}$, and a modular form of weight k and level N is a section of $\omega^{\otimes k}$ holomorphic at the cusps. It is possible to re-write this definition so that it makes sense over any ring R containing $1/N$. Briefly, a modular form f of weight k and level N over R is a rule assigning to each triple (A,η,κ) consisting of an elliptic curve A over $\operatorname{Spec}R'$, a basis η for $\omega_{A/R'}$, and a level structure κ, an element of R'; here R' is an R-algebra. When we apply f to the Tate curve and its canonical differential over $R[[q]]$, then the element of $R[[q]]$ that we obtain is the Fourier series of f.

For the final approach, one computes the formal completion at a cusp of the compactification of S_N. It is the formal spectrum of a power series ring over $\mathbb{C}$ in one variable. By extending the modular form f to the compactification, and using the computation, one obtains the Fourier series of f.

2. The analytic definition of Fourier-Jacobi series.

Piatetski-Shapiro (1966) (especially §12, §15) associates a Fourier-Jacobi series with any automorphic form (or function) on a Siegel domain. In order to apply the construction to an automorphic form f on a bounded symmetric domain D, we use the realization of D as a Siegel domain of the third kind corresponding to a rational boundary component F of X (see V.2). The Fourier-Jacobi series attached to f and the boundary component F is then of the form

$$FJ^F(f) = \sum_{\rho} \psi_\rho(u,t)e^{2\pi i(\rho,z)}.$$

Here ρ runs over a finitely generated abelian group, t runs over the symmetric Hermitian domain F, and, for a fixed ρ and t, $\psi_\rho(u,t)$ is a theta function. Recall that a theta function can be regarded as a section of a line bundle on an abelian variety. Since a mixed Shimura variety is, roughly speaking, a sum of line bundles (with the zero sections removed) over an abelian scheme over a Shimura variety, a function on it can be written $(\psi_\rho(u,t))_\rho$ where t is a point of the Shimura variety and $\psi_\rho(u,t)$ is a section of the line bundle indexed by ρ on the abelian variety over t. The similarity of two expressions is not a coincidence.

3. The modular definition of Fourier-Jacobi series.
There is a very complete discussion of Fourier-Jacobi series for Siegel modular forms in Chai and Faltings (1989), and a briefer discussion for automorphic forms on a Shimura variety of Hodge type in Brylinski (1983), §5.

4. A formal-algebraic definition of Fourier-Jacobi series.
Let (G,X) be a pair defining a Shimura variety, and let $W^F_\cdot$ be a Cayley filtration on G. In (VI.1.11) above, we derived from these data a triple $(P,W.,Y)$ defining a mixed Shimura variety. Let K be a compact open subgroup of $G(\mathbb{A}_f)$, and let $\Gamma = G(\mathbb{Q}) \cap K$ and $\Gamma_P = P(\mathbb{Q}) \cap K$. Then $W_{-2}P(\mathbb{C})$ contains a canonical self-adjoint homogeneous cone C. Choose a $\bar{\Gamma}(F)$-admissible fan Δ in C. Then we can form the partial compactification $\mathrm{Sh}_K(G,X)_\Delta$ of $\mathrm{Sh}_K(G,X)$ along F. Assume that $\mathrm{Sh}_K(G,X)_\Delta$ is smooth, and that the boundary of $\mathrm{Sh}_K(G,X)$ in it is a divisor with normal crossings. We then write $\mathrm{Sh}_K(G,X)_{\hat{\Delta}}$ for the formal completion of $\mathrm{Sh}_K(G,X)$ along the boundary. We can also form the partial compactification $M_{K_P}(P,W.,Y)_\Delta$ of $M_{K_P}(P,W.,Y)$, and the formal completion $M_{K_P}(P,W.,Y)_{\hat{\Delta}}$ of $M_{K_P}(P,W.,Y)$ along its boundary in $M_{K_P}(P,W.,Y)_\Delta$.

CONJECTURE 4.1.. *There is a canonical isomorphism*

$$\mathrm{Sh}_K(G,X)_{\hat{\Delta}} \to G_\ell(\mathbb{Z})\backslash M_{K_P}(P,W.,Y)_{\hat{\Delta}}$$

The isomorphism should correspond to the isomorphism on the level of analytic spaces.

The statement should be regarded as giving a precise description of the structure of $\mathrm{Sh}(G,X)$ near the boundary component F. For Siegel modular varieties, it is proved in Chai and Faltings (1989), IV.

A $G_{\mathbb{C}}$-equivariant vector bundle $\mathcal{J}$ on $\check{X}$, defines automorphic vector bundles $\mathcal{V}(\mathcal{J})$ and $\mathcal{V}_M(\mathcal{J})$ on $\mathrm{Sh}(G,X)$ and $M(P,W.,Y)$ respectively; extend the vector bundles to the partial compactifications; the isomorphism in (3.1) will give an isomorphism of the formal completions: $\mathcal{V}(\mathcal{J})_{\hat{\Delta}} \approx \mathcal{V}_M(\mathcal{J})_{\hat{\Delta}}$. A section f of $\mathcal{V}(\mathcal{J})$ will extend to a section of $\mathcal{V}(\mathcal{J})_{\Delta}$, and map to a section of $FJ^F(f)$ of $\mathcal{V}_M(\mathcal{J})_{\hat{\Delta}}$— this is the *Fourier-Jacobi series of f along F.*

5. Conjugates of Fourier-Jacobi series.

The map $f \mapsto FJ^F(f)$ should be compatible with the various maps $\phi^*_{\tau,x}$ (see V.5.1 and VI.4). The q-expansion principle should then allow us to deduce that an automorphic form is rational over a field L if and only if its Fourier-Jacobi series are.

Note that for noncompact Shimura varieties, this will give another description of the canonical model of minimal compactification: it is the Proj of the graded ring generated by automorphic forms whose Fourier-Jacobi series have coefficients in the reflex field. We mention that Baily and Karel have been attempting to give a totally different approach to some of the results in this article by directly constructing automorphic forms whose Fourier-Jacobi series are rational (in a suitable sense) over $E(G,X)$ and then showing that the Proj of the graded ring they define is the canonical model of the Shimura variety (see for example Baily (1985) and Karel (1986)).

6. Automorphic forms of half-integral weight.

Just as modular forms of half-integral weight for GL_2 correspond in a natural way to automorphic forms of integral weight on the mixed Shimura variety defined in (2.3) (see Eichler and Zagier 1985), so should all automorphic forms of half-integral weight on a Shimura variety correspond to automorphic forms of integral weight on a mixed Shimura variety.

Notes. There is an enormous literature on Fourier-Jacobi series. Apart from those referred to in the text, the following papers are most closely related to the main theme of this Chapter: Shimura

(1978b), (1978c); Garrett (1981), (1983); and Harris (1986). I understand that Richard Pink's Bonn thesis will examine the question of the formal-algebraic definition of Fourier-Jacobi series.

REFERENCES

Addington, S., *Equivariant holomorphic maps of symmetric domains*, Duke Math. J. **55** (1987), 65–88.

Ash, A., Mumford, D., Rapoport, M., and Tai, Y., "Smooth compactification of locally symmetric varieties," Math. Sci. Press, Brookline, 1975.

Baily, W., *Arithmetic Hilbert modular forms*, Revista Matemática Iberoamericana **1** (1985), 85-119.

Baily, W., and Borel, A., *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. Math. **84** (1966), 442–528.

Berndt, R., *Sur l'arithmétique du corps des fonctions elliptiques de niveau N*, Séminaire de Théorie des Nombres de Paris (1982-83), 21–33.

Blasius, D., *On the critical values of Hecke L-series*, Annals of Math. **124** (1986), 23–63.

Blasius, D.. Paper in preparation (1989).

Borel, A., *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem*, J. Differ. Geom. **6** (1972), 543–560.

Borel, A., et al., "Algebraic D-modules," Academic Press, Boston, 1987.

Borovoi, M., *Langlands's conjecture concerning conjugation of Shimura varieties*, Sel. Math. Sov. **3** (1983/84), 3–39.

Borovoi, M., *On the groups of points of a semisimple group over a totally real field,*, Problems in Group Theory and Homological Algebra, Yaroslavl, (1987), 142–149.

Bourbaki, N., "Topologie Général, Livre III," Hermann, Paris, 1960.

Brylinski, J.L., *Algébricité du quotient par un groupe discret d'un domaine de Siegel de 3ème espèce, provenant d'un espace hermitien symétrique*, manuscript, 1979.

Brylinski, J.L., *"1-motifs" et formes automorphes (Théorie arithmétiques des domaines de Siegel)*, in "Journées Automorphes," Publ. Math. Univ. Paris VII, 15, 1983, pp. 43–106.

Chai, C-L., "Compactification of Siegel Moduli Schemes," Lecture Note Series 107, London Math. Soc., London, 1985.

Chai, C-L., and Faltings, G., "Semiabelian Degeneration and Compactification," manuscript, 1989.

Clozel, L., *On the limit multiplicities of discrete series representations in spaces of automorphic forms*, Invent. math. **83**, 265–284.

Deligne, P., "Equations Différentielles á Points Singuliers Réguliers," Lecture Notes in Math. 163, Springer, Heidelberg, 1970.

Deligne, P., *Théorie de Hodge I*, in "Actes du Congrés International des Math. (Nice, 1970), 1," Gauthier-Villars, 1971a, pp. 425–430.

Deligne, P., *Théorie de Hodge II*, Publ. Math. I.H.E.S. **40** (1971b), 5–58.

Deligne, P., *Travaux de Shimura*, Sém. Bourbaki, Fév 1971, Exposé 389, Lect. Notes Math. vol. 244, Springer, Heidelberg, (1971c).

Deligne, P., *La conjecture de Weil pour les surfaces K3*, Invent. math. **15** (1972), 206–226.

Deligne, P., *Espaces hermitiens symétriques*, Notes from a seminar at IHES, Spring 1973.

Deligne, P., *Théorie de Hodge III*, Publ. Math. I.H.E.S. **44** (1975), 5–77.

Deligne, P., *Variétés de Shimura: interpretation modulaire, et techniques de construction de modéles canoniqes*, Proc. Symp. Pure Math., A.M.S. **33, part 2** (1979), 247–290.

Deligne, P.. Letter to Tate, 8th October, 1981 (reproduced in Lang 1983, VII.4).

Deligne, P., *Hodge cycles on abelian varieties*, (Notes by J.S. Milne), in "Hodge Cycles, Motives, and Shimura Varieties," Lect. Notes Math., vol. 900, Springer, Heidelberg, 1982a, pp. 9–100.

Deligne, P., *Motifs et groupes de Taniyama*, in "Hodge Cycles, Motives, and Shimura Varieties," Lect. Notes Math., vol. 900, Springer, Heidelberg, 1982b, pp. pp 261–279.

Deligne, P., *Catégories Tannakiennes*, preprint, 1989.

Deligne, P., and Milne, J., *Tannakian categories*, in "Hodge Cycles, Motives, and Shimura Varieties," Lect. Notes Math., vol. 900, Springer, Heidelberg, 1982, pp. 101–228.

Eichler, M., and Zagier, D., "The Theory of Jacobi Forms," Progress in Math., 55, Birkhäuser, Boston, 1985.

Faltings, G., *Arithmetic varieties and rigidity*, in "Séminaire de Théorie des Nombres, Paris 1982-83," Progress in Math., 51, Birkhäuser, Boston, 1984, pp. 63–77.

Faltings, G., *Arthmetische Kompaktifizierung des Modulraums der abelschen Varietäten*, in "Lecture Notes in Math. 1111," Springer, Heidelberg, 1985, pp. 321–383.

Garrett, P., *Arithmetic properties of Fourier-Jacobi series*, Amer. J. Math. **103** (1981), 1103–1134.

Garrett, P., *Modular curves on arithmetic quotients*, Duke Math. Jour. **49** (1982), 633–654.

Garret, P., *Arithmetic structure of automorphic forms on bounded symmetric domains*, Amer. J. Math. **105** (1983), 1171–1194, 1195–1216.

Garrett, P., *Imbedded modular curves and arithmetic of automorphic forms on bounded symmetric domains*, Duke Math. Jour. **51** (1984), 431–458.

van der Geer, G., "Hilbert Modular Surfaces," Springer, Heidelberg, 1988.

Harris, M., *Arithmetic vector bundles and automorphic forms on Shimura varieties, I*, Invent. math. **82** (1985), 151–189.

Harris, M., *Arithmetic vector bundles and automorphic forms on Shimura varieties, II*, Compositio Math **60** (1986), 323–378.

Harris, M., *Functorial properties of toroidal compactifications of locally symmetric varieties*, Proc. L.M.S. (to appear) (1989).

Hartshorne, R., "Ample Subvarieties of Algebraic Varieties," Lect. Notes in Math., vol. 156, Springer, Heidelberg, 1970.

Helgason, S., "Differential Geometry, Lie Groups, and Symmetric Spaces," Academic Press, New York, 1978.

Iitaka, S., "Algebraic Geometry, An Introduction to Birational Geometry of Algebraic Varieties," Springer, Heidelberg, 1982.

Karel, M., *Special points on Hilbert modular varieties*, Revista Matemática **2** (1986), 367–380.

Katz, N., *p-adic properties of modular schemes and modular forms*, in "Modular Functions of One Variable III," Lecture Notes in Math. 350, Springer, Heidelberg, 1973, pp. 69–190.

Kazhdan, D., *On arithmetic varieties II.*, Isr. J. Math. **44** (1983), 139–159.

Kempf, G., Knudsen, F., Mumford, D., and Saint-Donat, B., "Toroidal Embeddings, I.," Lect. Notes in Math., 339, Springer, Heidelberg, 1972.

Knutson, D., "Algebraic Spaces," Lect. Notes in Math., 203, Springer, Heidelberg, 1971.

Kobayashi, S., and Nomizu, K., "Foundations of Differential Geometry I," Wiley, New York, 1963.

Lang, S., "Complex Multiplication," Springer, Heidelberg, 1983.

Langlands, R., *Automorphic representations, Shimura varieties, and motives, Ein Märchen*, in "Automorphic Forms, Representations, and L-functions," Proc. Symp. Pure Math. 33, Part 2, AMS, Providence, 1979 pages 205–246.

Margulis, G., *Discrete groups of motions of manifolds of nonpositive curvature*, Amer. Math. Soc. Transl. **109** (1977), 33–45.

Milne, J., "Étale Cohomology," Princeton U. P., Princeton, 1980.

Milne, J., *Abelian varieties with complex multiplication (for pedestrians)*, manuscript, 19th Sept., 1981.

Milne, J., *The action of an automorphism of* $\mathbb{C}$ *on a Shimura variety and its special points*, Prog. in Math. 35, Birkhäuser, 1983, 239–265..

Milne, J., *Automorphic vector bundles on connected Shimura varieties*, Invent. math. **92** (1988), 91–128.

Milne, J., and Shih, K-y., *The action of complex conjugation on a Shimura variety*, Annals of Math. **113** (1981a), 569–599.

Milne, J., and Shih, K-y., *Automorphism groups of Shimura varieties and reciprocity laws*, American J. Math. **103** (1981b), 911–935.

Milne, J., and Shih, K-y., *Langlands's construction of the Taniyama group*, in "Hodge Cycles, Motives, and Shimura Varieties," Lect. Notes in Math. 900, Springer, Heidelberg, 1982a, pp. 229–260.

Milne, J., and Shih, K-y., *Conjugates of Shimura varieties*, in "Hodge Cycles, Motives, and Shimura Varieties," Lect. Notes in Math. 900, Springer, Heidelberg, 1982b, pp. 280–356.

Miyake, K., *Models of certain automorphic function fields*, Acta Math. **126** (1971), 245–307.

Mumford, D., "Geometric Invariant Theory," Springer, Heidelberg, 1965.

Mumford, D., *Families of abelian varieties*, in "Algebraic Groups and Discontinuous Subgroups," Proc. Symp. Pure Math., 9, 1966, pp. 347–351.

Mumford, D., "Abelian Varieties," Oxford Univ. Press, Oxford, 1970.

Mumford, D., *An analytic construction of degenerating abelian varieties over complete rings*, Compositio Mathematica **24** (1972), 239–272.

Mumford, D., *A new approach to compactifying locally symmetric varieties*, in "Discrete Subgroups of Lie Groups," Oxford Univ. Press, Oxford, 1975, pp. 211–224.

Mumford, D., *Hirzebruch's proportionality theorem in the non-compact case*, Invent. Math. **42** (1977), 239–272.

Nadel, A., and Tsuji, H., *Compactification of complete Kähler manifolds of negative Ricci curvature*, J. Diffl. Geometry **28** (1988), 503–512.

Namikawa, Y., *A new compactification of the Siegel space and degeneration of abelian varieties*, I, Math. Ann. 221 (1976), 97–141; II, ibid., 201–241.

Namikawa, Y., "Toroidal Compactification of Siegel Spaces," Lect. Notes in Math. 812, Springer, Heidelberg, 1979.

Oda, T., "On Torus Embeddings and Applications," Tata Lecture Notes, 57, 1978.

Oda, T., "Convex Bodies and Algebraic Geometry," Springer, Heidelberg, 1987.

Petri, M., *Arithmetic classification of Kuga fiber varieties of quaternion type*, Duke Math. J. **58** (1989), 469–498.

Piatetskii-Shapiro, I., "Géométrie des Domaines Classiques et Théorie des Fonctions Automorphes," Dunod, Paris, 1966.

Piatetskii-Shapiro I., *Interrelations between the Tate and Hodge hypotheses for Abelian varieties*, Math. USSR Sbornik 14 (1971), 615–625.

Saavedra Rivano, N., "Catégories Tannakiennes," Lect. Notes in Math. 265, Springer, Heidelberg, 1972.

Satake, I., *Holomorphic imbeddings of symmetric domains into a Siegel space*, Amer. J. Math. **87** (1965), 425–461.

Satake, I., *On the arithmetic of tube domains (Blowing-up of the point at infinity)*, Bull. Amer. Math. Soc. **79** (1973), 1076–1094.

Satake, I., "Algebraic Structures on Symmetric Domains," Princeton Univ. Press, Princeton, 1980.

Shappacher, N., "On the Periods of Hecke Characters," Springer, Heidelberg, 1988.

Serre, J-P., *Sur les groupes de Galois attachés aux groupes p-divisibles*, in "Proceedings of a Conference on Local Fields," Springer, Heidelberg, 1967, pp. 118–31.

Serre, J-P., "Abelian l-adic Representations and Elliptic Curves," Benjamin, New York, 1968.

Shih, K-y., *Anti-holomorphic automorphisms of arithmetic automorphic function fields*, Annals of Math **103** (1976), 81–102.

Shih, K-y., *Existence of certain canonical models*, Duke Math. J. **45** (1978), 63–66.

Shimura, G., *Number fields and zeta functions associated with discontinuous groups and algebraic varieties*, in "Proc. Intern. Congress Math.," Mir, Moscow, 1968, pp. 290–299.

Shimura, G., *On canonical models of arithmetic quotients of bounded symmetric domains*, Annals of Math **91** (1970), 141–222, 528–549.

Shimura, G., *On arithmetic automorphic functions*, in "Actes, Congrès Int. Math. 1970," Vol 2, Gauthier-Villars, Paris, 1971a, pp. 343–348.

Shimura, G., "Introduction to the Arithmetic Theory of Automorphic Functions," Princeton Univ. Press, Princeton, 1971b.

Shimura, G., *On some problems of algebraicity*, in "Proc. Int. Cong. Math. 1978a, Vol I, 373–379."

Shimura, G., *On certain reciprocity-laws for theta functions*, Acta Math. **141** (1978b), 31–71.

Shimura, G., *The arithmetic of automorphic forms with respect to the unitary group*, Ann. of Math. **107** (1978c), 569–605.

Shimura, G., *Automorphic forms and the periods of abelian varieties*, J. Math. Soc. Japan **31** (1979), 561–592.

Shimura, G., *The arithmetic of certain zeta functions and automorphic forms on orthogonal groups*, Ann. of Math. **111** (1980), 313–375.

Shimura, G., *On certain zeta functions attached to two Hilbert modular forms*, Ann. of Math. **114** (1981), 127–164, 569–607.

Tannaka, T., *Dualität der nicht-kommutativen Gruppen*, Tôhoku Math. J. **53** (1938), 1–12.

Tate, J., *Number theoretic background*, in "Automorphic Forms, Representations, and L-functions," Proc. Symp. Pure Math, Vol 33, Part 2, AMS, Providence, 1979, pp. 3–26.

Tate, J.. Unpublished manuscript on Complex Multiplication (1981).

Wolf, J., and Korányi, A., *Generalized Cayley transformations of bounded symmetric domains*, Amer. J. Math. **87** (1965), 899–939.

Mathematics Department, University of Michigan, Ann Arbor, MI 48109, USA.